

Stress Components in Cubical Elements with Reference to the Coordinate System

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Abstract

In this paper, stress analysis of a body subjected to external forces is presented with regards to different planes of the system. Cubical elements are the main topic of analysis here on which normal and shearing stresses are tested. The normal stress on a principal plane is also called the principal stress. The principal stress in a coordinate system of a particular orientation and with a specific positioning is studied.

Keywords: Stress Components, Cubical Elements, 3D Mohr's circle representation

INTRODUCTION

The stress components on a cubical element may be useful for describing fundamental relations with reference to the coordinate system. The cubical element is one of building blocks constituting the elastic body.

Figure 1.1 (a) shows a body subjected to normal uniform stress distribution. The

body is assumed to consist of infinite number of cubical elements. Figure 1.1 (b) shows one of cubes, representing a point in the body, in which nine stress components are used to describe a stress state in terms of location, magnitude and direction

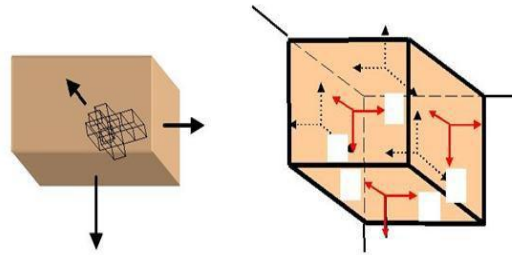


Figure 1.1 (a) A body subjected to uniform stress; and (b) one of cubes in ‘(a)’ subjected to

$$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \text{ or } \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

Or

The first subscript of each stress component indicates plane and the second direction. The nine stress components can be reduced to six components because some of stress components are equal. This can be found by taking the summation of the moments (ΣM) about z-axis, y-axis and x-axis:

$\Sigma M_z = 0$ for z- axis,

$$\tau_{xy}(\Delta y \Delta z) \Delta x = \tau_{yx}(\Delta x \Delta z) \Delta y \quad (1.1)$$

and therefore $\tau_{xy} = \tau_{yx}$.

Similarly for x and y axes, $\tau_{yz} = \tau_{zy}$ and $\tau_{zx} = \tau_{xz}$. Consequently, the state of stress at a point can be now described by six components: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$.

RELATION OF PRINCIPAL STRESS WITH OTHER STRESS COMPONENTS

When a body subjected to external forces, a range of different planes may considered for stress analysis. The planes where no shearing stresses but normal stresses exist are called the principal planes. The normal stress on each principal plane is referred to as the principal stress. Figure 1.2 shows the principal stress on area JKL as a result of choosing the coordinate system in a particular orientation and for a particular position.

The areas on the tetrahedron are found in relation with A or area JKL:

The principal stress (σ) in Figure 1.2 may be related with the stress components by taking the summation of the forces in x, y and z directions:

$$\Sigma F_x = 0,$$

$$(\sigma - \sigma_x)Al - \tau_{yx}Am - \tau_{zx}An = 0 \quad (1.2a)$$

$$\Sigma F_y = 0,$$

$$-\tau_{xy}Al + (\sigma - \sigma_y)Am - \tau_{zy}An = 0 \quad (1.2b)$$

$$\Sigma F_z = 0,$$

$$-\tau_{xz}Al - \tau_{yz}Am + (\sigma - \sigma_z)An = 0 \quad (1.2c)$$

These three equations are compacted for relations between the principal stresses and other stress components:

$$\begin{bmatrix} (\sigma - \sigma_x) & -\tau_{yx} & -\tau_{zx} \\ -\tau_{xy} & (\sigma - \sigma_y) & -\tau_{zy} \\ -\tau_{xz} & -\tau_{yz} & (\sigma - \sigma_z) \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \quad (1.2d)$$

The direction cosines l , m and n can be eliminated from the three equations to find an expression for σ .

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \quad (1.2e)$$

where

$$I_1 = \sigma_x + \sigma_y + \sigma_z \quad (1.2f)$$

As seen in Equation (1.2e), I_1 , I_2 and I_3 are not functions of direction cosines. As such, they are independent of the coordinate system location and therefore they are called the invariants.

STRESSES ON OBLIQUE PLANE

Any other planes than the principal planes may be called the oblique planes in which always shear stress exists when subjected to external forces [Figure 1.3 (a)]. The total stress (S) on the oblique plane can be

resolved into three components (S_x , S_y , and S_z) [Figure 1.3 (b)] and

Taking the summation of the forces in the x, y and z directions yields:

$$S^2 = S_x^2 + S_y^2 + S_z^2 \quad (1.3a)$$

$$S_x = \sigma_x l + \tau_{yx} m + \tau_{zx} n$$

$$S_y = \tau_{xy} l + \sigma_y m + \tau_{zy} n \quad (1.3b)$$

$$S_z = \tau_{xz} l + \tau_{yz} m + \sigma_z n$$

The normal stress (σ_n) may be found in terms of S_x , S_y and S_z [Figure 1.3 (c)] by projecting the total stress components (S_x , S_y , and S_z) onto the normal stress direction:

$$\sigma_n = S_x l + S_y m + S_z n \quad (1.3c)$$

$$\text{Also, } S^2 = \sigma_n^2 + \tau^2 \quad (1.3d)$$

Therefore, the shear stress is found as a function of principle (normal) stresses (σ_1 , σ_2 and σ_3):

$$\tau^2 = (\sigma_1 - \sigma_2)^2 l^2 m^2 + (\sigma_1 - \sigma_3)^2 l^2 n^2 + (\sigma_2 - \sigma_3)^2 m^2 n^2 \quad (1.4)$$

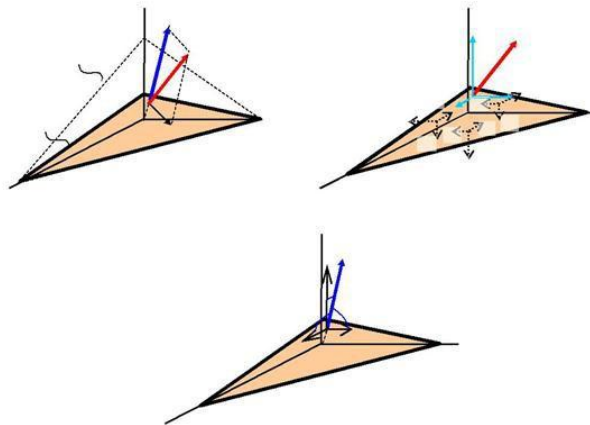


Figure 1.3 Stress components on oblique plane: (a) total stress, S , consisting of normal stress (σ_n) and shear stress (τ);

(b) total stress components (S_x , S_y , and S_z) in x, y, and z directions; and (c) normal stress components of the total stress can be obtained by projecting total stress components onto the normal stress direction

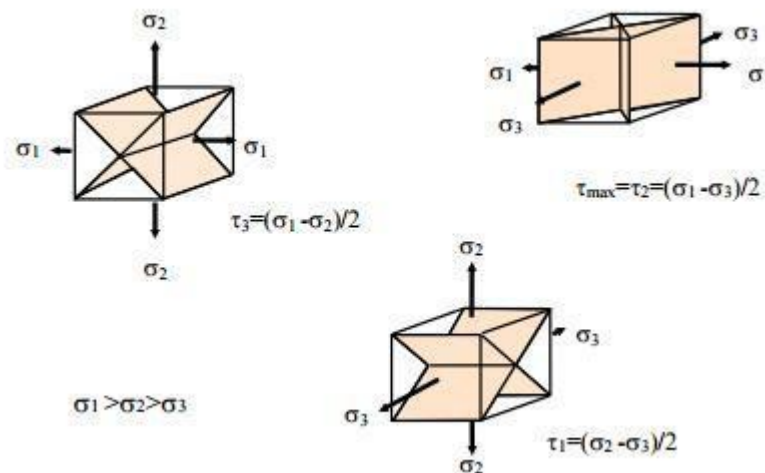


Figure 1.4 The maximum shear planes at an angle of 45° with the three principal axes.

The maximum shear stress criterion (or Tresca yield criterion) assumes that yielding occurs when the maximum shear stress ($\tau_{\max} = \sigma_1 - \sigma_3$) reaches its yielding point. In the case of uni-axial loading, the maximum principle stress (σ_1) reaches its yielding point (σ_{ys}) so that, and the maximum shear stress becomes:

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_1 - 0}{2} = \frac{\sigma_{ys}}{2} \quad (1.6)$$

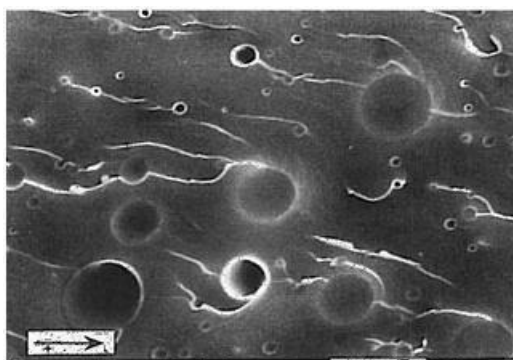


Figure 1.5 Rubber modified epoxy showing cavities on fracture surface.

The arrow indicates fracture propagation direction and the bar represents 10 μm . [After Kim and Ma, 1996]

In general, the deformation of an element consists of volume and shape changes. The volume change is a result of proportional change in element edge lengths. In contrast, the shape change is a result of disproportional change in element edge lengths as well as element corner angle change. The former is associated with volumetric modulus (K) and hydrostatic stress (or mean stress) while the latter is associated with shear modulus

(G) and shear stress. For example, the hydrostatic stress creates cavities during deformation as shown in Figure 1.5 or increases the brittleness while shear stress contributes to the material flow. The total stress for deformation consists of hydrostatic stress and stress deviator i.e.

Total stress = Hydrostatic (or mean) stress (σ_m) + Stress deviator.

The *hydrostatic* stress or mean stress is defined as

$$\sigma_m = \frac{I_1}{3} = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \quad (1.7)$$

and the stress deviator can be found by subtracting the hydrostatic stress from the total stress:

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} - \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} =$$

$$\begin{bmatrix} \frac{2\sigma_x - \sigma_y - \sigma_z}{3} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \frac{2\sigma_y - \sigma_x - \sigma_z}{3} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \frac{2\sigma_z - \sigma_x - \sigma_y}{3} \end{bmatrix} \quad (1.8)$$

This relation is graphically shown in **Figure 1.6**.

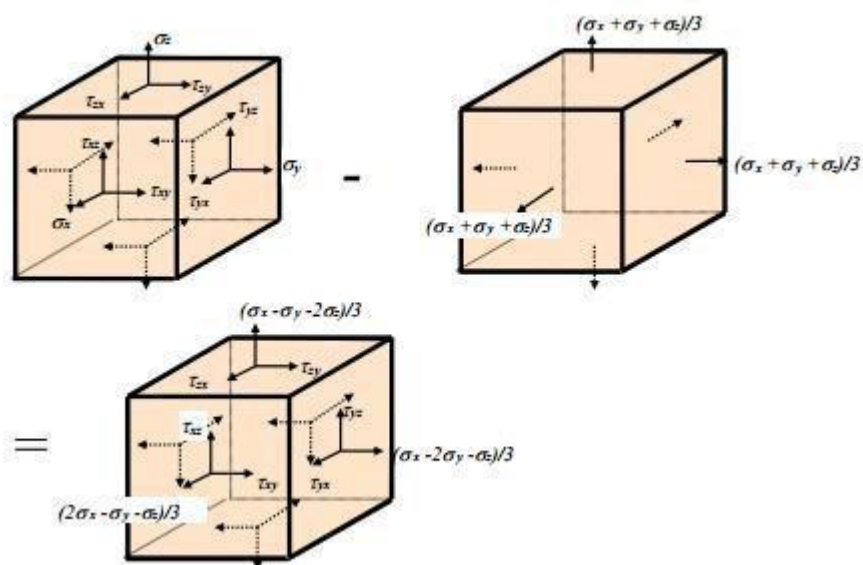


Figure 1.6 Superposition of stress components.

It can easily be shown that the stress deviator involves the principal shear stresses. For example,

$$\sigma'_x = \frac{2\sigma_x - \sigma_y - \sigma_z}{3} \quad (1.9)$$

If we choose principal stresses in the equation, the stress deviator becomes a function of principal shear stresses,

$$\sigma'_1 = \frac{2\sigma_1 - \sigma_2 - \sigma_3}{3} = \frac{2(\sigma_1 - \sigma_2)}{3} + \frac{(\sigma_1 - \sigma_3)}{2} = \frac{2(\tau_3 + \tau_2)}{3} \quad (1.10)$$

Similarly, (1.11)

$$\begin{aligned} \sigma'_2 &= \frac{2\sigma_2 - \sigma_1 - \sigma_3}{3} = \frac{2(\sigma_2 - \sigma_1)}{3} + \frac{(\sigma_2 - \sigma_3)}{2} = \frac{2(-\tau_3 + \tau_1)}{3} \\ \sigma'_3 &= \frac{2\sigma_3 - \sigma_1 - \sigma_2}{3} = \frac{2(\sigma_3 - \sigma_1)}{3} + \frac{(\sigma_3 - \sigma_2)}{2} = \frac{-2}{3}(\tau_2 + \tau_1) \end{aligned} \quad (1.12)$$

where τ_1 , τ_2 and τ_3 are the principal shear stresses.

3D MOHR'S CIRCLE REPRESENTATION

The three principal stresses with the maximum shear stresses (τ_1 , τ_2 and τ_3) can graphically be represented as shown in Figure 1.7. The radius of each circle represents the maximum (or principle) shear stress. Accordingly, Equation (1.5) can be found from the Mohr's circle. Figure 1.7 (a) shows a case of uni-axial tensile loading (s_1) with If s_3 varies from zero to -50MPa (compressive stress), the principal shear stresses increase and becomes a state of stress where

hydrostatic stress is zero as given in Figure 1.7 (b), resulting in more chances for material flow or high ductility than that of the case in Figure 1.7 (a) because of the increase in shear stress. Figure 1.7 (c) is the limiting case where the three circles reduces to a point where 50MPa and the three principle shear stresses are zero. In this case, no material ductility is possible in the absence of stress deviator. Therefore, it is theoretically possible to have a stress state where the hydrostatic stress component exists without the stress deviator. Examples for the state of stress

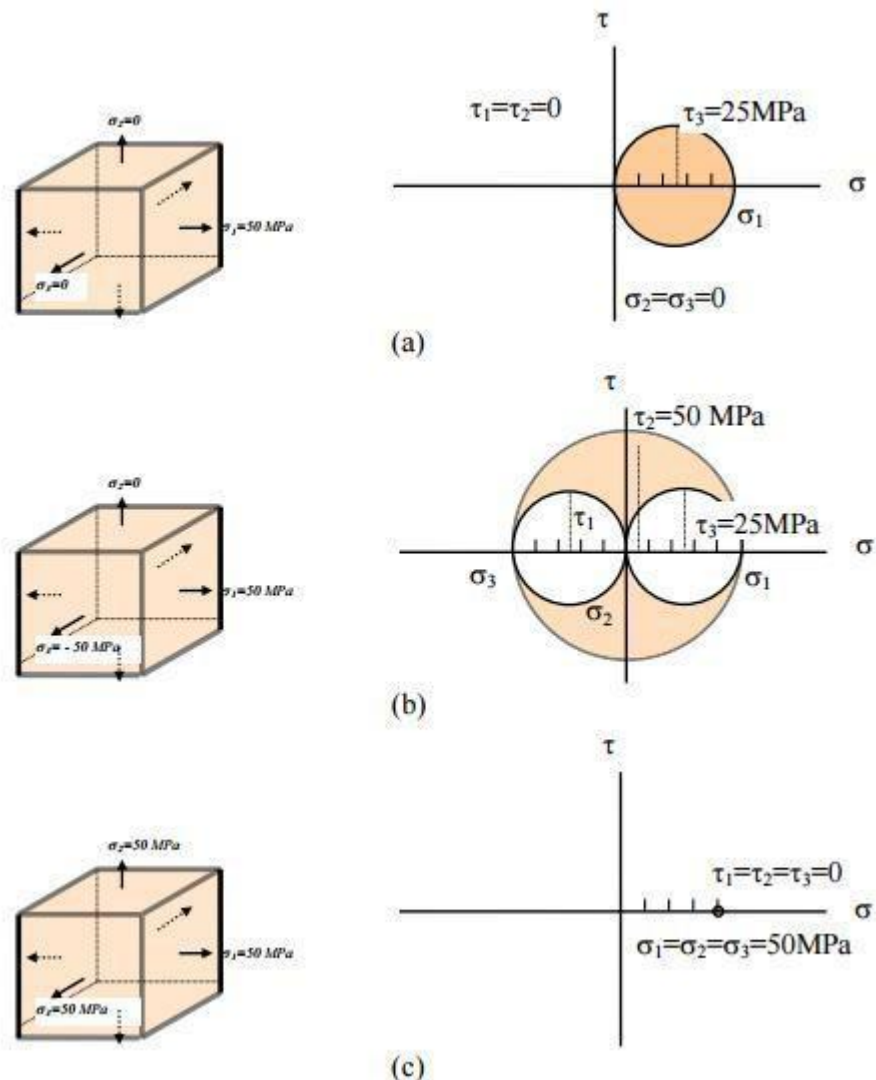


Figure 1.7 Various states of stress on elemental cube and Mohr's circles; (a) uni-axial tension; (b) tension and compression without hydrostatic stress; and (c) hydrostatic stress without shear stress.

where relatively large principal shear stresses are involved are found in various processes in metal forming (e.g. wire drawing through a die) involving lateral compressive stresses and a longitudinal tensile stress. Also, some localized

deformation of reinforcing particles on fracture surfaces of advanced materials is

caused by such a state of stress involving large shear stresses as shown in Figure 1.8.

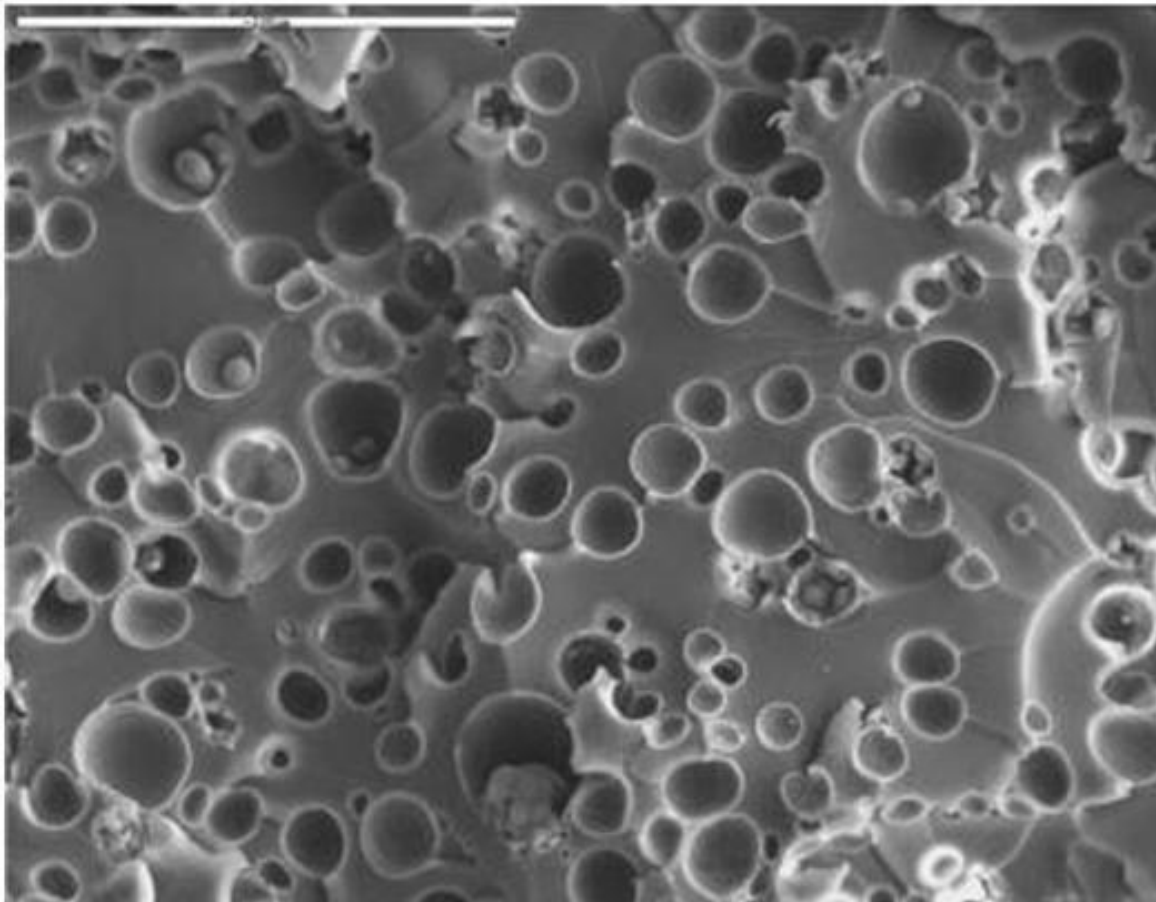


Figure 1.8 Fracture surface of hollow microsphere reinforced epoxy under plane strain in the vicinity of initial crack tip. Each hollow-microsphere experienced a tensile stress in the direction perpendicular to the fracture surface and simultaneously lateral compressive stresses. The crack propagation direction is from top to bottom. The scale bar represents 100 μm . [After Kim, 2007]²

STRAIN AT A POINT

As previously discussed, the deformation is due to the volume and shape change. We need to define the displacement components. When a point P moves to P' in coordinates x, y, z as shown in Figure 1.9, respective u, v and w are called displacement components. To find a general form of strain, let us consider first

the length change using an element subjected to a load in the x -direction as shown in Figure 1.10.

When the load is applied, the solid line becomes the dashed line. Accordingly, A moves to A' and B moves to B' and the x direction normal strain of the infinitesimal segment is given by

$$e_x = e_x = \frac{A'B' - AB}{AB} = \frac{dx + \frac{\partial u}{\partial x} dx - dx}{dx} = \frac{\partial u}{\partial x}. \quad (1.13a)$$

Similarly, the displacement derivatives for y and z directions can be found:

$$e_{yy} = \frac{\partial v}{\partial y}, \quad e_{zz} = \frac{\partial w}{\partial z}. \quad (1.13b)$$

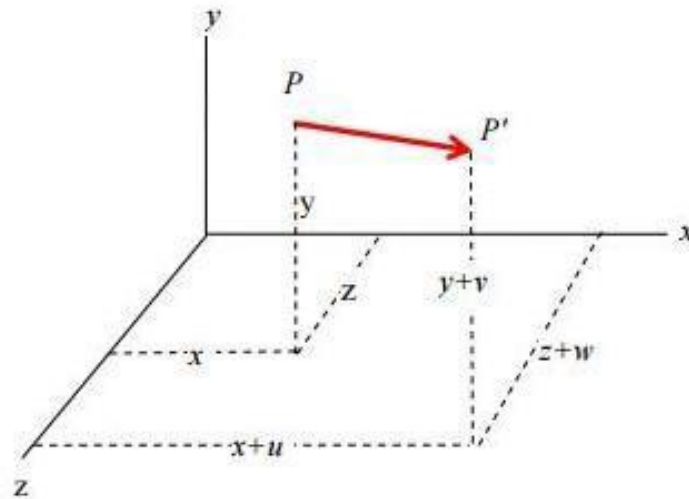


Figure 1.9 Displacement of a Point P.

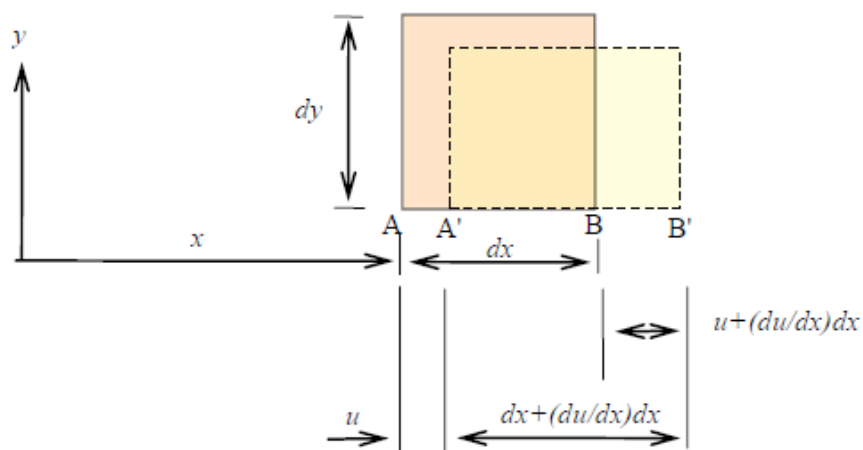


Figure 1.10 Deformation in the x-direction

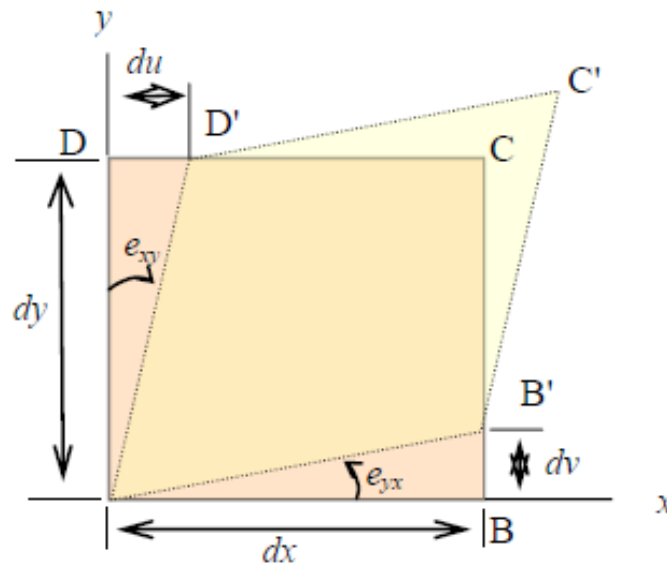


Figure 1.11 Angular Distortion of an Element

For further displacement derivatives, let us consider a shape change using an element in the xy plane, which is subjected to shear deformation as shown in Figure 1.11. The

element has undergone an angular distortion and thus the angular displacement derivatives along the x and y axes are given by

$$e_{yx} = \frac{BB'}{AB} = \frac{\partial v}{\partial x} \quad (1.13c)$$

and

$$e_{xy} = \frac{DD'}{DA} = \frac{\partial u}{\partial y} \quad (1.13d)$$

respectively. Similarly, the rest of components can be found:

$$e_{ij} = \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \quad (1.13e)$$

In general, components such as e_{xy} , e_{yx} , etc., other than those for length change produce both *shear strain and rigid-body rotation*.

For example, those given in **Figure 1.12**

(a) represent a pure rotation with an average of $\frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$ and Figure 1.12

(b) a pure rotation with an average of

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

Thus, the rigid body rotation components (w_{ij}) can be identified as:

$$w_{ij} = \begin{bmatrix} w_{xx} & w_{xy} & w_{xz} \\ w_{yx} & w_{yy} & w_{yz} \\ w_{zx} & w_{zy} & w_{zz} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial u}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial z}\right) & 0 \end{bmatrix}. \quad (1.14)$$

Accordingly, the strain components (w_{ij}) can be found by subtracting rigid body rotation components (ϵ_{ij}) from the displacement derivatives:

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial z} \end{bmatrix}. \quad (1.15)$$

For $w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$, $\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ short, and are called the strain tensor and

the *rotation tensor* respectively. Also,

$$e_{ij} = \varepsilon_{ij} + \omega_{ij} \quad (1.16)$$

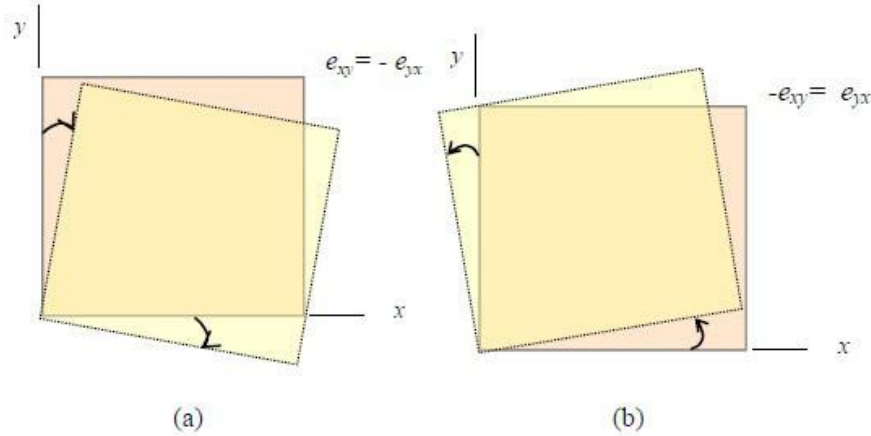


Figure 1.12 Pure rotation without shear.

Referring to **Figure 1.13**, engineering shear strain components (γ_{ij}) are defined as

$$\gamma_{xy} = e_{xy} + e_{yx} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}, \quad \gamma_{xz} = 2\varepsilon_{xz}, \quad \gamma_{yz} = 2\varepsilon_{yz} \quad (1.17)$$

In summary, strain components are

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad (1.18)$$

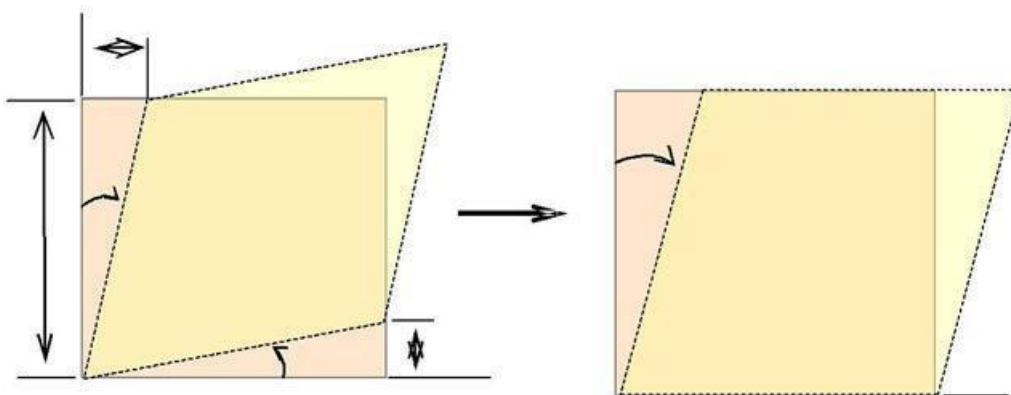


Figure 1.13 Engineering shear strain γ .

The volume strain (Δ) (see Figure 1.14) is defined as

$$\Delta = \frac{\text{Final volume} - \text{Original volume}}{\text{Original volume}}$$

or

$$\begin{aligned} \Delta &= \frac{(1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z)dx dy dz - dx dy dz}{dx dy dz} \\ &= (1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z) - 1 \\ &\approx \varepsilon_x + \varepsilon_y + \varepsilon_z \end{aligned} \quad (1.19)$$

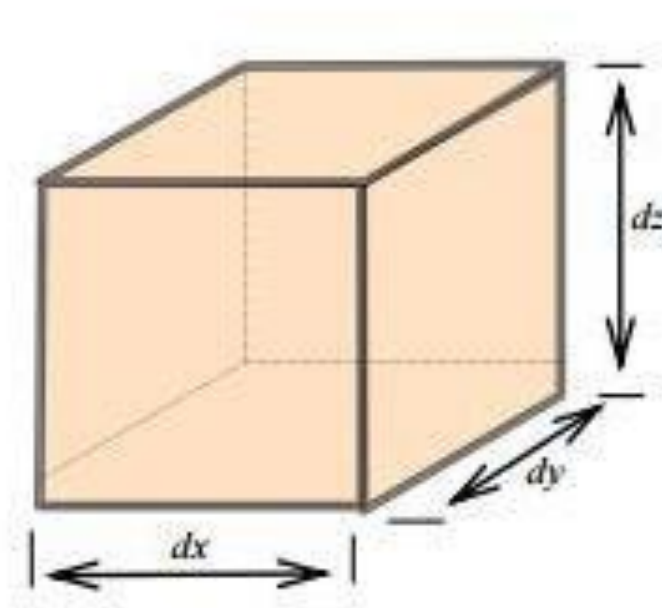


Figure 1.14 An Elemental Cube

The mean strain or the hydrostatic component of strain, which contributes to volume change, is also defined as

$$\varepsilon_m = \frac{\varepsilon_x + \varepsilon_y + \varepsilon_z}{3} = \frac{\varepsilon_{kk}}{3} = \frac{\Delta}{3} \quad (1.20)$$

Then, the *strain deviator* (ε'_{ij}) which contributes to shape change, can be obtained by subtracting from ε_m each of the normal strain components:

$$\varepsilon'_{ij} = \begin{bmatrix} \varepsilon_x - \varepsilon_m & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y - \varepsilon_m & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z - \varepsilon_m \end{bmatrix} = \begin{bmatrix} \frac{2\varepsilon_x - \varepsilon_y - \varepsilon_z}{3} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \frac{2\varepsilon_y - \varepsilon_z - \varepsilon_x}{3} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \frac{2\varepsilon_z - \varepsilon_x - \varepsilon_y}{3} \end{bmatrix} \quad (1.21)$$

In complete analogy between stress and strain equations, the principal strains are the roots of the cubic equation:

$$\varepsilon^3 - I_1 \varepsilon^2 + I_2 \varepsilon - I_3 = 0 \quad (1.22)$$

where

$$\begin{aligned} I_1 &= \varepsilon_x + \varepsilon_y + \varepsilon_z \\ I_2 &= \varepsilon_x \varepsilon_y + \varepsilon_y \varepsilon_z + \varepsilon_z \varepsilon_x - \frac{1}{4}(\gamma_{xy}^2 + \gamma_{zx}^2 + \gamma_{yz}^2) \\ I_3 &= \varepsilon_x \varepsilon_y \varepsilon_z + \frac{1}{4} \gamma_{xy} \gamma_{zx} \gamma_{yz} - \frac{1}{4}(\varepsilon_x \gamma_{xy}^2 + \varepsilon_y \gamma_{zx}^2 + \varepsilon_z \gamma_{yz}^2). \end{aligned} \quad (1.23)$$

As already discussed for stress, I_1 , I_2 and I_3 are not functions of direction cosines. As such, they are independent of the coordinate system location and therefore they are called 'invariants'.

Also, the principal (engineering) shear strains are

$$\begin{aligned} \gamma_1 &= \varepsilon_2 - \varepsilon_3, \\ \gamma_2 &= \varepsilon_1 - \varepsilon_3 (= \gamma_{\max}) \\ \gamma_3 &= \varepsilon_1 - \varepsilon_2. \end{aligned} \quad (1.24)$$

CONCLUSION

After perusal of this paper, researchers and students will be able to understand the impact of stress on a cubical object with regards to different planes of a system. This will be an important step for understanding of mechanics which eases us into advanced level studies of different types of stresses and strains on various objects.

The study of principal stresses in different orientations with regards to a coordinate system and specific positioning develops our understanding of stress-strain analysis to that is important for building a fail-safe design of structural components.

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