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Symmetric Pair from Symmetric Space and Lie group

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Abstract

A Pseudo-Riemannian manifold whose group of symmetry contains inversion symmetry about every point is called a symmetric space in differential geometry. On the other hand, in Riemannian geometry, a complete, simply connected Riemannian manifold whose curvature tensor is invariant under parallel transport is called a symmetric space. This paper is the study of the formation of symmetric pair from symmetric space. Also, a explanation is given how a symmetric pair can be formed from Lie group.

Keywords: *Lie group, Lie algebra, isotropy group, cartan decomposition, transvections*

I. INTRODUCTION

The study of symmetric space was initiated by Elie Cartan and employs many of the sophisticated tools of modern mathematics such as differential geometry, group theory, topology, quotient space, Lie group and Lie algebras. For different points of view symmetric spaces are different. The most significant symmetric space are Riemannian symmetric space which is also a Riemannian manifold. Riemannian symmetric space has a great importance in various branches of Mathematics such as Grassmannians and bounded symmetric domains.

If considering the point of view of Lie theory, then a symmetric space is the quotient space G/H of a Lie group G where H is Lie subgroup and $\mathfrak{h}$ is the Lie algebra of H which is required to be the $+1$ eigenspace of an involution of the Lie algebra $\mathfrak{g}$ of G.

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2. SYMMETRIC SPACE

This section is about the notion of symmetric space at the view point of Riemannian manifold. Firstly, there is a review about geodesically complete Riemannian manifold and Hopf-Rinow theorem. Lastly, there is also a discussion about the related theorem of symmetric space.

A Riemannian manifold (P, l) is said to be geodesically complete if for all $a \in P$ the exponential map Exp_a is defined on all T_aP that is all geodesic $\gamma \in P$ can be defined for all $t \in \mathbb{R}$. A normal ball $B_\delta(x)$ at $x \in P$ on a Riemannian manifold (P, l) is the image of an open ball $\breve{B}_\delta(x)$ in the tangent space T_xP of the exponential map Exp_x. We can define the infimum of all curves on P from x to y by the distance $d(x,y)$ where $x, y \in P$. It can be shown that the distance function is continuous and that geodesics locally minimize the distance between two points and also if a curve is a minimum then it must be a geodesic.

Theorem-1[5]: Hopf-Rinow Theorem: Let, (P, l) be a Riemannian manifold. If (P, l) is geodesically complete, then any two point $x, y \in P$ can be connected by a geodesic.

Proof: Let, P is geodesically complete then the exponential map Exp_x is defined for all T_xP. Also, let, $x, y \in P$, $r = d(x,y)$ and $B_\delta(x)$ be a normal ball in P, whose boundary is $S_\delta(x)$. Let, y_0 be a point where the continuous function $d(y,k)$, $k \in S_\delta(x)$ takes on it's infimum. Then, $y_0 = Exp_x \delta v$, where $v \in T_xP$ and $|v| = 1$. Let, λ be a geodesic given by $\lambda(t) = Exp_x tv$.

Now, $D = \{t \in [0,r] : d(\lambda(t), y) = r - t\}$, then we will show that $r \in D$ and $\lambda(r) = y$. Now, since $0 \in D$, so D is non-empty and by the continuity of the distance D is closed. It remains to show that if $t_0 \in D$ and $t_0 < r$, then there exist $\delta > 0$ such that $t_0 + \delta \in D$. Since, D is closed, thus $\sup D = r$ and $r \in D$. So, we can write $d(\lambda(t), y) = r - r = 0$, that is $\lambda(r) = y$. Assuming at $\lambda(t_0)$, $B_{\delta'}(\lambda(t_0))$ be a normal ball and S' be the boundary of $B_{\delta'}(\lambda(t_0))$.

Also, suppose that, y_0' be the point where, $d(m, y)$, $m \in S'$ takes on its minimum.

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Thus,

$d(\lambda(t_0), y) = \delta' + \min\{d(m,y) : m \in S'\} = \delta' + d(y_0', y)$

. But, $d(\lambda(t_0), y) = r - t_0$. So, $r - t_0 = \delta' + d(y_0', y)$.

Now, if, $y_0' = \lambda(t_0 + \delta')$, then $r - t_0 = d(\lambda(t_0 + \delta'), y)$, thus $t_0 + \delta' \in D$. Now, we have to prove that $y_0' = \lambda(t_0 + \delta')$. From the triangle inequality

$d(x, y_0') \geq d(x,y) = d(y, y_0') = r - (r - t_0 - \delta') = t_0 + \delta'$

, which is obvious, but the broken curve joining x and y_0' by going along λ from x to $\lambda(t_0)$ and along the exponential ray from $\lambda(t_0)$ to y_0' in $B_{\delta'}(\lambda(t_0))$ has length $t_0 + \delta'$. So, it can write $d(x, y_0') = t_0 + \delta'$. Now, the broken curve must be a geodesic because of the minimalism of the distance. But as it is starting out as λ it must remain λ, so $y_0' = \lambda(t_0 + \delta')$.

Hence the proved

Now, by using mathematical terminology, A symmetric space (P, l) is a Riemannian manifold if for every point $x \in P$ there exist an isometry σ_x of (P, l) such that $\sigma_x(x) = x$ and $d\sigma_x = -id_{T_xP}$. This type of isometry is called an involution at $x \in P$.

Lemma-2[6]: If (P, l) be a symmetric space and $\sigma_x : (P, l) \to (P, l)$ be an involution at $x \in P$. Then, $\sigma_x(\lambda(t)) = \lambda(-t)$ for all geodesic $\lambda \in P$ such that $\lambda(0) = x$.

Proof: Let, $\lambda(0)$ and $\dot{\lambda}(0)$ be the initial data, then there is a uniquely define geodesic $\lambda : J \to P$. But, both the geodesic $t \mapsto \sigma_x(\lambda(t))$ and $t \mapsto \lambda(-t)$ take the value $\lambda(0)$ and its tangent is $\dot{\lambda}(0)$ at $t = 0$.

Lemma-3[6]: Let, (P, l) be a symmetric space and $\lambda : J \to P$ be the geodesic which satisfy $\lambda(0) = x$ and $\lambda(m) = y$, then $\sigma_y \circ \sigma_x(\lambda(t)) = \lambda(t + 2m)$. If $v \in T_{\lambda(t)}P$, $d\sigma_x(d\sigma_x(v)) \in T_{\lambda(t+2m)}P$ is the vector at $\lambda(t + 2m)$, which is obtained by parallel transport of v along λ.

Proof: Let, $\check{\lambda}(t) = \lambda(t + m)$, then with $\check{\lambda}(0) = y$, $\check{\lambda}$ is a geodesic.

Now, from *Lemma-2*, it can write,

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$\sigma_y\left(\sigma_x(\lambda(t))\right) = \sigma_y(\lambda(-t)) =$
$\sigma_y\left(\check{\lambda}(-t-m)\right) = \check{\lambda}(t+m) = \lambda(t+2m)$
. Now, $d\sigma_x(w)$ is parallel if $v \in T_xP$ and w is a parallel vector field along λ with $w(x) = x$.

So,

$d\sigma_y \circ d\sigma_x\left(w(\lambda(t))\right) = w(\lambda(t+2m))$.

Corollary-4[6]: Since every symmetric space (P, l) is geodesically complete, so it can say that any two points $x, y \in P$ in the same path component of P can be connected by a geodesic.

3. ISOMETRY GROUP

Let, (P, l) be a symmetric space of isometry group $I(P)$ and H be the connected component of $I(P)$ which contains the neutral element $e \in H$ that is H is the identity component of $I(P)$. Also, suppose that N_{x_0} be the isotropy group of H at $x_0 \in P$. Now, we prove a very important theorem related to isometry group.

Theorem-5[6]: If (P, l) be a symmetric space and H be the identity component of $I(P)$, then H acts transitively on P.

Proof: From *corollary-4* we know that any $x, y \in P$ can be connected by a geodesic λ. Now, let, $\lambda(0) = x$ and $\lambda(t) = y$, then $y = \tau_{tv}(x)$, since τ_{tv} is the family of translations along λ. Thus, the action is transitive.

Lemma-6[8]: Let, $h: J \to I(P)$ be a map such that $h(j)x$ depends smoothly on (j, x), where J is a smooth manifold, $j \in J$ and $x \in M$ contained in a neighbourhood of $x_0 \in P$. Then for all $x \in P$ the dependence is smooth.

Proof: We can cover P with intersecting balls from x_0 to an arbitrary point in P, if we prove the theorem for $x \in P$ in a normal ball around $y \in P$. Now, let, $x \in B_\delta(y)$, then there exist a $v \in T_yP$ such that $x = Exp_y v$ and v depends smoothly on x.

So,

$h(j)(x) = h(j)Exp_x v =$
$Exp_{h(j)(x)}\left(dh(j)(x)\right)v$
, where the right side smoothly depends on (j, x).

Now, we show charts and actions on manifold are smooth by following lemma.

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Lemma-7[8]: If (P, l) be symmetric space with involutions σ_x, transvections τ_v and N_{x_0} be the isotropy group of the isometries $I(P)$ acting at a point $x_0 \in P$, then

(a) $\sigma_x(y)$ depends smoothly on $(x, y) \in P \times P$.

(b) $\tau_x(y)$ depends smoothly on $(v, y) \in T_x P \times P$.

(c) $\tau_x(y)$ depends smoothly on $(x, y) \in P \times P$ such that $\tau_v(x_0) = x$.

(d) $n(y)$ depends smoothly on $(n, y) \in N_{x_0} \times P$.

Following theorem is about how the quotient of the isometry group and isotropy group can be identified with the symmetric space.

Theorem-8[8]: Let, (P, l) be symmetric space and $H = I(P)$ act transitively on P. Now, if N be the isotropy group of H at $x_0 \in P$, then the map $\varphi: H/N \to P$ with $hN \mapsto h(x_0)$ is a diffeomorphism such that $\varphi \circ \alpha(h) = h \circ \varphi$, where $\alpha(h')hN = h'(hN) = h'hN$.

Proof: Here, clearly φ is well defined and bijective since $v_n(x_0) = x_0$ for all $n \in N$. So,

$$\varphi \circ \alpha(h')hN = \varphi(h'hN) = v_{h'h}(x_0) = v_{h'}v_h(x_0) = v_{h'}(\varphi(hN))$$

Now, φ is smooth because $\varphi \circ \pi(h) = v_h$ and π is also smooth. We show that φ^{-1} is also smooth. This can be proved by showing that $(H/N) \geq \dim P$. Then we show $d\varphi$ is injective, so $d\varphi$ is isomorphism and φ^{-1} is locally smooth if we apply inverse function theorem. But φ^{-1} exist globally so by gluing lemma φ^{-1} is smooth.

Consider, $P = \bigcup_{i=1}^{\infty} U_i$, where U_i are the images of the charts on H/N. By Baire Category Theorem at least one $\overline{U}_i$ contains an open subset of P because P is locally compact. Since H/N acts smoothly thus the open subset on H has a bijective correspondence with open set W in H/N. If we use the charts we get an open map $\mathbb{R}^n \longrightarrow W \longrightarrow U_i \longrightarrow \mathbb{R}^m$, which is onto map. So we can not have $n < m$ that is $\dim(H/N) \geq \dim P$. So, It remains to show that $\dim(H/N) \leq \dim P$ by proving injectivity. Thus, $\dim(H/N) = \dim P$. But, $\varphi \circ \alpha(h) = v_h \circ \varphi$, thus, $\varphi = v_h \circ \varphi \circ \alpha(h^{-1})$. If we differentiate at h', we get,

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$(d\varphi)_{h'n} = (dv_h \circ d\varphi)_{h^{-1}hn} = (d\alpha(h^{-1}))_{h'n}$

.

If we choose h in such a way that $h^{-1}h'N = bN$ in $\varphi = v_h \circ \varphi \circ \alpha(h^{-1})$, then $d\varphi$ is injective at $h'N$ if it is injective at bN. Now, we show the injectivity at eN. Let, $v \in T_{eN}(H/N)$ with $d\varphi(v) = 0$. Now, there exist a $X \in \mathfrak{h}$ such that $d\pi(X) = v$ since $d\pi$ is onto and has kernel $\mathfrak{k}$, where $\mathfrak{h}$ is the Lie algebra of H. The curve $\lambda(t) = v_{\exp(tX)}(x_0)$ satisfies

$$\frac{d\lambda(t)}{dt} = \frac{d}{dt}(v_{\exp(tX)}(x_0)$$
$$= \frac{d}{dt}(\varphi \circ \pi(\exp(tX))$$
$$= d\varphi \circ d\pi_{\exp(tX)}(X)$$
$$= d\varphi \circ d\pi \circ dE_{\exp(tX)}(X(e))$$
$$= d\varphi \circ d\mu(\exp(tX)) \circ d\pi(X(e))$$
$$= dv_{\exp(tX)} \circ d\varphi \circ d\pi(X(e))$$
$$= dv_{\exp(tX)} \circ d\varphi(v) = 0$$

So, $\lambda \in N$ and $X \in \mathfrak{k}$ that is $\lambda = 0$.

4. SYMMETRIC PAIRS

Let, H be a Lie group, N be a closed subgroup of H and k_{x_0} an involutive automorphism on H, then a pair (H, N) is said to be a Riemannian symmetric pair if the following conditions are satisfied

(a) $(H_{k_{x_0}})_0 \subseteq N \subseteq H_{k_{x_0}}$.

(b) Adjoint of N is a compact subset of $HL(h)$.

Where $HL(h)$ denotes the Lie group of all bijective linear maps on $\mathfrak{h}$ and $H_{k_{x_0}}$ is defined by $H_{k_{x_0}} = \{h \in H: k_{x_0} = h\}$ that is $H_{k_{x_0}}$ are the elements of H that are left invariant by k_{x_0}.

Now, for the formation of symmetric pair from symmetric space as before, let, H, N_{x_0} be the identity component of the isometry and isotropy groups on P. Then by *Theorem-8*, clearly, there is a bijective correspondence $H/N \leftrightarrow P$, $hN \mapsto h(x_0)$.

Consider the mapping $k_{x_0}: H \longrightarrow H$ such that

$k_{x_0}(h) = \alpha_{x_0} \circ h \circ \alpha_{x_0} = \alpha_{x_0} \circ h \circ \alpha_{x_0}^{-1}$,

where α_{x_0} be an involution. Then, $k_{x_0}^2 = id$ and $k_{x_0}(h)x = \alpha_{x_0} \circ h \circ \alpha_{x_0}^{-1}(x)$.

Thus, k_{x_0} is an involutive Lie group automorphism of H. Also $(dk_{x_0})_e: \mathfrak{h} \longrightarrow \mathfrak{h}$ is a involutive Lie algebra automorphism. Let, $H_{k_{x_0}} = \{h \in H: k_{x_0}h = h\}$ and $\mathfrak{h}_{k_{x_0}}$ be its Lie algebra. Then, $H_{k_{x_0}}$ is a Lie group since it is closed. From the

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following theorem we can construct a symmetric pair from symmetric space.

Theorem-9[8]: Let, (P, l) be a symmetric space where x_0 is a fixed point. If H be the identity component of isometry group and N be the isotropy group of H at x_0. Then, the map $H/N \to P$ with $N \mapsto h(x_0)$ is a bijective map. If k_{x_0} is an involutive automorphism of H with stabilizer $H_{k_{x_0}}$ then $k_{x_0}(h) = \alpha_{x_0} \circ h \circ \alpha_{x_0}$ such that $(H_{k_{x_0}})_0 \subseteq N \subseteq H_{k_{x_0}}$, which says that (H, N) is a symmetric pair because $Ad(N)$ is compact, N is closed and bounded and also Ad is a homeomorphism.

Proof: We prove the last statement. If $X \in \mathfrak{h}_{x_0}$, then from the definition of Lie algebra, clearly, $\exp(tX) \in H_{k_{x_0}}$, then, we write, $\exp(tX) = k_{x_0}(\exp(tX))$, therefore $dk_{x_0} = X$. Now, since, $dk_{x_0} = X$, thus $\exp(tX) = \exp(tdk_{x_0}) = k_{x_0}(\exp(tX))$. So, Lie algebra $\mathfrak{h}_{x_0} = \{X \in h: dk_{x_0}X = X\}$. Now, let, $n \in N$ then,

$k_{x_0}(n) = \alpha_{x_0} \circ n \circ \alpha_{x_0}(x_0) = x_0 = n(x_0)$

and

$dk_{x_0}(n) = d\alpha_{x_0} \circ dn_{x_0} \circ d\alpha_{x_0}(x_0) = \left(-id_{T_{x_0}P}\right) \circ dn_{x_0} \circ \left(-id_{T_{x_0}P}\right) = dn_{x_0}$

.

This shows that $k_{x_0}(n) = n$ that is $n \in H_{x_0}$, thus $N \subseteq H_{x_0}$. Since, $X \in \mathfrak{h}_{x_0}$, then

$\alpha_{x_0}((\exp(tX))x_0) = \alpha_{x_0}(\exp(tX))\alpha_{x_0}(x_0)$

, so $(\exp(tX))x_0$ is a fixed point in α_{x_0}.

For small t, $(\exp(tX))x_0$ is close to x_0, thus $\exp(tX) = x_0$. Therefore, $\exp(tX) \in N_0 = N$ and $X \in \mathfrak{k}$ =kernal of $d\alpha$. So $\mathfrak{h}_{x_0} \subseteq \mathfrak{k} \Longrightarrow (H_{x_0})_0 \subseteq N$. Hence the proved.

Now, we explain how a symmetric pair is formed from a Lie group and a closed subgroup which is the basis of following lemma and theorem.

Lemma-10[8]: Let, H be a Lie group and N be a closed subgroup of H. If $n \in N$, then consider the notation $In\ h(h') = hh'h^{-1}$. Then the statement $\mu(h) \circ \pi = \pi \circ In\ n$ is true and by differentiating it, at $d\mu(h)_{eN} \circ d\pi_e = d\pi_e \circ Ad\ n$, where, $\mu(h')hN = (h')(hN) = h'hN$.

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Proof: Let, $h \in H$ and $n \in N$, then by definition of μ, $\mu(n)(hN) = nhN = nhn^{-1}N = In(n)hN = \pi(In(n)h)$.

Thus, $\mu(n) \circ \pi = \pi \circ In\, n$.

Theorem-11[8]: Let, H be a Lie group and N be a closed subgroup of H with identity components H_0, N_0. Let, μ be the action which is defined by $\mu(h')hn = h'hn$ and H/N be the quotient manifold. Consider, $I \in Z(N) \subseteq N$ such that

$\mu(I)eN = eNd\pi(I)_{eN} = -id_{H/N}$. (1)

Then for every compact subset N' in $N, N_0 \subseteq N' \subseteq N \cap H_0$.

From above assumption a Riemannian symmetric pair (H_0, N') is formed, where, $k_{eN}(h) = IhI^{-1} = In(I)h$. Assumption (1) is equivalent to

$X + Ad\ I(X) \in \mathfrak{k}$, for all $X \in \mathfrak{h}$. (2)

Condition (2) holds if $\mathfrak{k}$ has a linear complement $\mathfrak{m}$ in $\mathfrak{h}$ that is $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{m}$ on which $Ad\ I = -id$.

Proof: Suppose, (1) holds, then by *Lemma-10*, $\mu(n) \circ \pi = \pi \circ In(n)$ and $d\mu(n)_{eN} \circ d\pi_e = d\pi_e \circ Ad\ n$. Now, putting, $n = I$, $-d\pi = d\pi \circ Ad\ I$. Thus, write, $d\pi(X + Ad(I)X) = 0$. So, $(X + Ad(I)X \in \mathfrak{k}$ =kernal of $d\pi$ for all $X \in \mathfrak{h}$.Thus, (1) implies (2). Now, let (2) holds. That is, $-d\pi_e = d\pi \circ Ad\ I = d\mu(I)_{eN} \circ d\pi_e$.

Clearly, $d\mu(I)_{eN} = -id_{H/N}$. Thus (2) implies (1). Now, it remains to show that we have a symmetric pair. Since, $I \in Z(N)$, $In\ I_{|N} = id_N$ and $Ad\ I_{|\mathfrak{k}} = id_{\mathfrak{k}}$. If, $X \in \mathfrak{m}$, where, $\mathfrak{m} \oplus \mathfrak{k} = \mathfrak{h}$. Then, $X + Ad(X) = F$, where, $F \in \mathfrak{k}$. But, F depends linearly on X, so $F = T(X)$, where, $T: \mathfrak{m} \longrightarrow \mathfrak{k}$ is a linear map. Thus, $(Ad\ I)^2X = Ad\ I\big(-X + T(X)\big) = -\big(-X + T(X)\big) = X$.

Since, $(Ad\ I_{|\mathfrak{k}})^2 = id_{\mathfrak{k}}$ and $Ad\ (I)^2 = id_{\mathfrak{h}}$. Thus, if $X \in \mathfrak{h}$ then $In\ (I)^2(\exp(X)) = \exp(dI_n(I)^2X) = \exp(Ad\ (I)^2X) = \exp(X)$

Thus, In is an involutive automorphism on H_0.

Now it remains to show that $((H_0)_{In\,I})_0 \subseteq N' \subseteq (H_0)_{In\,I}$. Since $(H_{In\,I})_0 = (H_0)_{In\,I}$, thus, $((H_0)_{In\,I})_0 = (H_{In\,I})_0$ and $(H_0)_{In\,I} = H_0 \cap H_{In\,I}$. So, $N' \subseteq (H_0)_{In\,I}$

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since $N' \subseteq H_0$, $N' \subseteq H_{In\,I}$ because $I \in Z(N)$ and $N' \subseteq N$. Secondly, if $\mathfrak{h}_{In\,I} \subseteq \mathfrak{k}' = \mathfrak{k}$ then $(H_{In\,I})_0 \subseteq N'$. Now, let, $X \in \mathfrak{h}_{In\,I}$ then $X = dIn(I)X = Ad\ I(X)$ and

$d\pi(X) = d\pi(Ad(I)X) = d\mu(I)d\pi(X) = d\pi(X)$

Thus, $d(X) = 0$ and $X \in \mathfrak{k}$. Finally, obviously, since Ad is a homeomorphism so $Ad\ N'$ is compact.

Proposition-12[9]: Let, (H, N) be a symmetric pair whose cartan decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{m}$. If, now, $\lambda(t) = \exp(tX)$, since $X \in \mathfrak{m}$. Then, x_0 is the geodesic in P with $\lambda(0) = x_0$ and $\dot{\lambda}(0) = X \in \mathfrak{m} \simeq T_{x_0}P$.

Proposition-13[9]: Let, (H, N) be a symmetric pair whose cartan decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{m}$. Then,

(a) $(\nabla_{P^*}Q)(x_0) = [P^*, Q](x_0)$ for any vector field Q on H/N.

(b)

$(S(P^*, Q^*)R^*)(x_0) = -[[P, Q], R]^*(x_0)$,

for $P, Q, R \in \mathfrak{m}$.

5. CONCLUSION

Let, $H = O(4), N = O(3)$, with $I \in Z(N)$, where, $O(n)$ is orthogonal group and

$$I = \begin{pmatrix} 1 & * & * \\ * & -1 & * \\ * & * & -1 \end{pmatrix}.$$

Now, $H/N = O(4)/O(3)$ and $X \in \mathfrak{o}(4)$, Lie algebra of $O(4)$.

Let, $X = \begin{pmatrix} R & Q \\ -Q^t & U \end{pmatrix}$.

So, $Ad\ I(X) = Ad\ I \begin{pmatrix} R & Q \\ -Q^t & U \end{pmatrix}$

$= \begin{pmatrix} R & -Q \\ Q^t & U \end{pmatrix}$.

And, $X + Ad\ I(X)$

$= \begin{pmatrix} R & Q \\ -Q^t & U \end{pmatrix} + \begin{pmatrix} R & -Q \\ Q^t & U \end{pmatrix}$

$= 2\begin{pmatrix} R & 0 \\ 0^t & U \end{pmatrix} \in \mathfrak{k}$.

Thus I satisfies (2) in *Theorem-11* and we get a symmetric pair $(SO(4), SO(3))$.

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