MANTECH Publications

Journal of Engineering Mathematics & Statistics
Volume 7, Issue 1, January-June 2023

On the Exponential Diophantine Equation 33x + (33u + Q22) = z22

V. S. Akilandeswari
Assistant Professor
Department of Mathematics
Saranathan College of Engineering, Trichy, India
Corresponding Authors' Email: - *akilasubbu22@gmail.com*

Abstract

In this paper, the all-possible solutions of the Exponential Diophantine equation $3^x + (3^u + \beta 2) = z^2$ *with* $u < x$, β *as non-negative integers, also with the condition that* β *is not divisible by 3 and* x, y, z *being non-negative integers, are found. The above Exponential Diophantine equation takes the solutions* $(u, \beta, 1,0,2)$, $(0,1,0,3,3)$, $(1,0,0,1,2)$, $(u, \beta, log_3(1 + 2(3^u + \beta^2)), 2,1 +(3^u + \beta^2))$, $(u, \beta, log_3 (3^u(2\beta - 1 + 3^u)), 1, \beta + 3^u)$, $(u, \beta, log_3(3^u(3^u - 2\beta - 1)), 1, 3^u - \beta)$ *for* u *and* β *taking values in such a way that* $log3(1 + 2(3^u + \beta^2))$, $log_3(3^u(2\beta - 1 + 3^u))$, $log_3(3^u(3^u - 2\beta - 1))$ *takes only all possible integer values. Some of the examples are given for the possible values of* u *and* β*. It is easily witnessed here that, the solutions for the equation with larger values of* u, β *can be obtained by using these results.*

Keywords: *Exponential Diophantine Equation (EDE), Integer solutions.*

INTRODUCTION

The general Exponential Diophantine equation (EDE) $a^x + b^y = z^2$ have been studied by a number of researchers [6,13,14]. The EDE $p^x + q^y = z^2$ was studied by Rabago [7,9] respectively in the years 2015 and 2013 by considering p, q as twin primes. In the year 2018, Burshetein[10] gave all the possible solutions of the EDE $p^x + (p + 4)^y = z^2$, when p, $p + 4$ are primes and $x + y = 2,3,4$. Chochaisthit [12], gave the solutions of $p^x + (p + 1) = z^2$ where p is a Mersenne prime in the year 2013. Suvarnamani[1] in the year 2011 gave all possible solutions of the EDE $2^x + p^y = z^2$ for the prime numbers 2,3, $2^{k+1} + 1$. Sroysang[2] in the year 2013 proved that (0,1,2), (3,0,3), (2,4,5) are the solutions of the EDE $2^x + 3^y = z^2$. Asthana and

Singh[11] showed that the non-negative integer solutions of the EDE $3^x + 13^y = z^2$ are (1,0,2), (1,1,4), (3,2,14), (5,1,6). Rabago[8] proved that $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ will have the possible pair of solutions as {(4,1,10), (1,0,2)} and {(2,1,10), (1,0,2)} respectively.In the year 2012, Sroysang[4,5] showed that the EDE $3x + 5y = z2$ and $3x + 17y = z2$ have only (1,0,2) as the only possible integer solution.

In this paper, a general EDE $3^x + (3^u + \beta^2) = z^2$ is considered for obtaining the integer solutions with the conditions that $u < x$ and β being integers such that β is chosen as a non-negative integer which is not divisible by 3. Examples are given by taking particular values to u and β. It is seen that this method will be very useful for the researchers who involve a higher base to the power y.

PRELIMINARIES

Lemma 1.1

The unique solution of the EDE $3x + 1 = z2$, where x and z are non-negative integers,is (1,2).

Proof

In the given EDE, $3^x + 1 = z^2$, when $x = 0$, $z^2 = 2$ where 2 is not a perfect square. Similarly if$z = 0$, then $3^x = -1$ which is impossible to find x. So the EDE has no solution if $x = 0$ and $z = 0$. The given EDE can be also written as $z2 - 1 = 3x$ which can further be rewritten as $(z + 1)(z - 1) = 3^x \Rightarrow z - 1 = 3^t$ and $z + 1 = 3^{x-t}$ for some non-negative integer $t < x$. On solving these equations, it is found that $3^{x-t} - 3^t = 2 \Rightarrow 3(^{3x-2t} - 1) = 2 \Rightarrow 3^t = 1 \Rightarrow t = 0$. Hence $3^x - 1 = 2 \Rightarrow x = 1$ and hence $z = 2$. Thus the solution of $3^x + 1 = z^2$ is (1,2).

MAIN RESULT Theorem

The integer solutions of the Exponential Diophantine Equation(EDE) $3^x + (3^u + \beta^2) = z^2$are

1) $(u, \beta, 1,0,2)$,
2) (0,1,0,3,3),
3) (1,0,0,1,2),
4) $(u, \beta, log_3(1 + 2(3^u + \beta^2)), 2,1 + (3^u + \beta^2))$,
5) $(u, \beta, log_3(3^u(2\beta - 1 + 3^u)), 1, \beta + 3^u)$,
6) $(u, \beta, log_3(3^u(3^u - 2\beta - 1)), 1, 3^u - \beta)$

for non-negative integers $u < x$ and β taking values in a such way that $log3(1 + 2(3u + \beta 2))$, $log3(3u(2\beta - 1 + 3u))$, $log3(3u(3u - 2\beta - 1))$ takes only integer values and β is not divisible by 3..

Proof

Case a) It is assumed first that $x = 0$, in the EDE,$3^x + (3^u + \beta^2) = z^2$ (1) Then (1) becomes $1 + (3^u + \beta^2) = z^2$. This can be written in the form that $(z - 1)(z + 1) = (3^u + \beta^2)$. It is possible to find a $0 \leq \delta\delta < y$ such that $(z - 1) = (3^u + \beta^2)$ and $(z + 1) = (3^u + \beta^2)-\delta\delta$.On subtracting these two equations, it is seen that $(3^u + \beta^2)\delta\delta[(3^u + \beta^2)^{y^{-2\delta\delta}} - 1] = 2$. On adding $(z - 1)$ and $(z + 1)$, it is clear that $2z = (3^u + \beta^2) + (3^u + \beta^2)-\delta\delta$.

Subcase i) It is possible to equate $(3^u + \beta^2) = 2$ and $(3^u + \beta^2)^{-2\delta\delta} - 1 = 1$. This implies $u = 0$, $\beta = 1$, $\delta\delta = 1$, $y = 3$ and $z = 3$. So, one of the solutions to (1) is (0,1,0,3,3).

Subcase ii) It can also be possible that $(3^u + \beta^2) = 1$ and $(3^u + \beta^2)^{-2\delta\delta - 1} = 2$. By solving these two equations the solution (1,0,0,1,2) is obtained.

Case b) If it is assumed that $y = 0$, in the EDE (1), it becomes $3^x + 1 = z^2$ and by Lemma 1.1, the solution $(u, \beta, 1,0,2)$ is obtained.

Case c) Now let y take only even values. Then $y = 2k$ is assumed for some positive integer k. Then (1) becomes $3^x + (3^u + \beta^2)^{2k} = z^2$. From this it is seen that $(z - (3^u + \beta^2))(z + (3^u + \beta^2)) = 3^x$. It is possible to find an integer $0 \leq \alpha < x$ such that $(z - (3^u + \beta^2)) = 3^\alpha$ and $(z + (3^u + \beta^2)) = 3^{x^- \alpha}$.Hence $2(3^u + \beta^2) = 3(3x-2\alpha - 1)$. Thus from this it is obtained that $\alpha = 0$ and so $3^x - 1 = 2(3^u + \beta^2)$.Since $k \geq 1$,the minimum value of $3^x = 1 + 2(3^u + \beta^2)$ and so $z = 1 + (3^u + \beta^2)$. Hence the solution becomes $(u, \beta, log^3(1 + 2(3^u + \beta^2)), 2,1 + (3^u + \beta^2))u, \beta$ taking values such that $log^3(1 + 2(3^u + \beta^2))$ is an integer.

Case d) Let $y = 2l + 1$; $l \geq 0$, is an integer. Let y take only odd values. It is also know that $z \equiv 0$ or $1\ (mod\ 4)$. Hence z takes only even values. So let $z = 2m$ for $m > 0$.

Hence (1) becomes $3^x + (3^u + \beta^2)^{2l+1} = 4m^2$. This can be reformatted as $3^x + (3^u + \beta^2)2(3^u + \beta^2) = 4m^2$. This implies $3^x + 3(3^u + \beta^2)2^l = 4m^2 - \beta^2(3^u + \beta^2)2^l$. Hence $3^x +$

$3(3^u + \beta^2)2^l = [2m-(3^u + \beta^2)^l][2m+\beta(3^u + \beta^2)^l]$. As $u < x$ $3(3^{x-u} + (3^u + \beta^2)^{2l}) = [2m-(3^u + \beta^2)l][2m+\beta(3^u + \beta^2)l]$ - (2) Subcase i) It is possible that $[2m-(3^u + \beta^2)l] = 3^u$ and $[2m+(3^u + \beta^2)l] = 3^{x-u} + (3^u + \beta^2)^{2l}$.

On subtracting these two equations, it is obtained that $(3^u + \beta^2)[2\beta - (3^u + \beta^2)] = 3^{x-u} - 3^u$ and on adding $4m = 3^{x-u} + 3^u + (3^u + \beta^2)^{2l}$. From the former equation it is seen that $l = 0$.

Hence 2 $\beta - 1 = 3^{x-u} - 3^u \Rightarrow x = log^3[3^u(2\beta - 1 + 3^u)]$ and $z = \beta + 3^u$.
Thus the solution to this case is $(u, \beta, log^3(3^u(2\beta - 1 + 3^u)), 1, \beta + 3^u)$ where u, β taking values such that $log^3(3^u(2\beta - 1 + 3^u))$ is an integer.

Subcase ii) It is also possible that $[2m-(3^u + \beta^2)l] = 3^{x-u} + (3^u + \beta^2)2l$and $[2m+(3^u + \beta^2)^l] = 3$.

On subtracting these two equations, it is obtained that $(3^u + \beta^2)[2\beta + (3^u + \beta^2)] = 3^u - 3^{x-u}$ and on adding $4m = 3^{x-u} + 3^u + (3^u + \beta^2)^{2l}$. From the former equation it is seen that $l = 0$.

Hence $2\beta + 1 = 3^u - 3^{x-u} \Rightarrow x = log^3[3^u(3^u - 2\beta - 1)]$ and $z = 3^u - \beta$.
Thus the solution to this case is $(u, \beta, log^3(3^u(3^u - 2\beta - 1)), 1, 3^u - \beta)$ where u, β taking values such that $log^3(3^u(3^u - 2\beta - 1))$ is an integer.

Examples

In this section, some EDE are considered and solutions of those EDE with respect to the theorem 2.1 are found.

Example 2.1.1

The solutions of the EDE $3^x + 13^y = z^2$ can be discussed as 13 can be written as (3^2+2^2). So, it is seen hat according to our EDE, $u = 2$ and $\beta = 2$. So, the solutions are (1,0,2), (3,2,14).

Example 2.1.2

The solutions of the EDE $3^x + 4^y = z^2$ can be discussed as 4 can be written as (3^1+1^2). So, it is seen that according to our EDE, $u = 1$ and $\beta = 1$. So, the solutions are (1,0,2), (2,2,5).

Example 2.1.3

The solutions of the EDE $3^x + 364^y = z2$ can be discussed as 364 can be written as (3^5+11^2). So, it is seen that according to our EDE, $u = 5$ and $\beta = 11$. So, the solutions are (1,0,2), (6,2,365). 364 can also be wriiten as (3^1+19^2), for this $u = 1$ and $\beta = 19$. Applying this the solutions to the EDE are (1,0,2), (6,2,365).

Example 2.1.4

The solutions to the EDE (1) when $u = 30$ and $\beta = 5$ are (1,0,2),(60,1, 205891132094649).

Example 2.1.5

The solutions to the EDE (1) when $u = 37$ and $\beta = 4$ are (1,0,2),(74,1, 450283905890997000).

Example 2.1.6

The solutions to the EDE (1) when $u = 40$ and $\beta = 7$ are (1,0,2), (80,1, 12157665459056900000).

Example 2.1.7

The solutions to the EDE (1) when $u = 27$ and $\beta = 14$ are (1,0,2).

Example 2.1.8

The solutions to the EDE (1) when $u = 31$ and $\beta = 11$ is (1,0,2),(62,1, 617673396283947).

Example 2.1.9

The solutions to the EDE (1) when $u = 34$ and $\beta = 34$ is (1,0,2), (68,1, 16677181699666600)

Example 2.1.10

The solutions to the EDE (1) when $u = 39$ and $\beta = 17$ is (1,0,2), (78,1,4052555153018980000).

CONCLUSION

In this paper, it is proved that (u, β, 1,0,2), (0,1,0,3,3), (1,0,0,1,2),(u, β, $log^3(1 + 2(3^u + \beta^2))$, 2,1 + $(3^u + \beta^2)$) , (u, β, $log^3(3^u(2\beta - 1 + 3^u))$, 1, $\beta + 3^u$), (u, β, $log^3(3^u(3^u - 2\beta - 1))$, 1, $3^u - \beta$) are the solutions of the EDE, $3^x + (3^u + \beta^2)^y = z^2$.

Provided $log^3(1 + 2(3^u + \beta^2))$, $log^3(3^u (2\beta - 1 + 3^u))$, $log^3(3^u (3^u - 2\beta - 1))$ are integers. It was considered that $u < x$ and β are non-negative integers with the fact that β is not divisible by 3. Also, the particular solutions of the EDE were brought forward in the examples by giving different values of u and β in all possible cases. It was found that this theorem 2.1 is very useful for higher values of u and β in the EDE $3^x + (3^u + \beta^2) = z^2$. Hence this theorem will be very useful for the researchers to find solutions to the EDE of this form with the higher base of y.

REFERENCES

1. A.Suvaranamani., Solutions of the Diophantine equations $2^x + p^y = z^2$., Int. J. Math. Sci. Appl., 1 (3).,1415-1419., 2011.
2. B.Sroysang., More on the Diophantine equation $2^x + 3^y = z^2$., Int. J. Pure Appl. Math.,84., 133-137., 2013.
3. B.Sroysang., More on the Diophantine equation $3^x + 85^y = z^2$., Int. J. Pure Appl. Math.,91 (1)., 131-134., 2014.
4. B.Sroysang., On the Diophantine equation $3^x + 17^y = z^2$., Int. J. Pure Appl. Math., 89 (1)., 111-114., 2013.
5. B.Sroysang., On the Diophantine equation $3^x + 5^y = z^2$., Int. J. Pure Appl. Math., 81., 605-608., 2012.
6. D. Acu., On the Diophantine equations of the type $a^x + b^y = c^z$, Gen. Math, 13 (1), 67- 72., 2005.
7. J.B. Bacani and J.F.T Rabago., The complete set of solutions of the Diophantine equation $p^x + q^y = z^2$ for twin primes p and q., Int. J. Pure Appl. Math., 104., 517-521., 2015.
8. J.F.T Rabago, On two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$, Int. J. Math. Sci. Comp., 3, 28-29, 2013.
9. 9.J.F.T Rabago., More on Diophantine equation $p^x + q^y = z^2$., Int. J. Math. Sci. Comp., 3., 15-16., 2013.

10. N.Burshtein., All the solutions of the Diophantine equation $p^x + (p + 4)^y = z^2$, when $p, p + 4$ are primes and $x + y = 2,3,4$., Annals of Pure and Applied Mathematics., 1., 241-244., 2018.
11. S. Asthana and M.M. Singh, On the Diophantine equation $3^x + 13^y = z^2$, Int. J. Pure Appl. Math., 114, 301-304, 2017.
12. S.Chochaisthit., On the Diophantine equation $p^x + (p + 1) = z^2$ where p is a Mersenne prime., Int. J. Pure Appl. Math., 88., 169-172., 2013.
13. T. Hadano., On the Diophantine equation $a^x = b^y + c^z$, Math. J. Okayama Univ., 19., 1- 53., 1976.
14. Z.Cao., A note on the Diophantine equation $a^x + b^y = c^z$., Acta Arith., XCI (1), 85-89., 1999.

