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The Numerical Study of the Zakharov Equation as a Model for Nonlinear Wave-Wave Interaction

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Abstract

The coupled 1D Zakharov equation is used as the model equation for wave-wave interaction in ionic media in this work. For the model equations, a finite difference technique is developed. A new six-point system is developed that is equal to the multi-symplectic integrator. The numerical simulation for the model equations is also offered.

Keywords*—Multi-symplectic scheme; Zakharov equation, Energy conservation; Six-point scheme*

INTRODUCTION

Wave-wave interaction is a significant physical and mathematical problem. In science and engineering, wave-wave contact or wave collisions are prevalent events for both solitary and non-solitary waves. A prominent branch of nonlinear wave interaction in ionic media is mathematically isolated wave collision. Its use is widespread in mathematics and physics, including nonlinear optics and plasma physics [1,11,12]. Much research has been conducted on interactions across a wide range of physical systems. Transmission, reflection, annihilation, entrapment, the formation of solitary waves, and even mutual spiralling have all been documented as interaction possibilities. However, there are several limits in their numerical simulations in order to maintain accuracy. Furthermore, they ignore numerous system features such as energy conservation, momentum conservation, and so on. The above-mentioned linked 1D nonlinear Schrödinger system has been solved analytically and quantitatively in several tries.

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Specification has recently been given to multi-symplectic geometry [2-4,8]. For some conservative PDEs, Bridge and Reich presented the notion of multi-symplectic integrators in the form of a finite difference scheme [5,9,10]. Theoretical investigations revealed that it is a strictly local concept that may be expressed using a finite difference technique. As a result, the multi-symplectic integrator has good local invariant conservation features [13,15,16]. Because the CNLS system has a multi-symplectic structure, we may use this method to create a multi-symplectic integrator in difference equations form. In this study, we discretize the system using finite difference techniques to demonstrate the CNLS system's multi-symplectic structure. We use numerical simulations to demonstrate the benefit of the CNLS system's multi-symplectic structure. In Section 2, we developed a six-point difference technique for linked nonlinear Schrödinger systems that is equal to a multi-symplectic integrator. The conservation feature of a linked nonlinear Schrödinger system is investigated in Section 3. Section 4 presents numerical simulations of the linked nonlinear Schrödinger system.

The Coupled Partial Differential Equations of Zakharov equation type is given by

$$iE_t + E_{xx} - nE = 0,$$
$$n_{tt} - n_{xx} - |E|_{xx}^{2} = 0$$

Here, *E(x,t)* is the slowly varying envelope of the high frequency field, and *n* is the density of the media or ions in media. These Zakharov equations can be approximated by Nonlinear Schrödinger Equation.

Where $E=p(x,t)+iq(x,y),\eta=\mu(x,t)+i\xi(x,y)$

We have

$$q_t - p_{xx} = (q\xi - p\mu)$$
$$p_t + q_{xx} = (p\xi + q\mu)$$
$$\mu_{tt} - \left((p + (q)^2\right) = 0$$

$$- \quad)^2$$

$$\mu$$

$$xx \qquad\qquad xx \qquad\qquad xx$$

$$\xi_{tt} - \xi_{xx} = 0$$

Introducing the canonical momenta

$$p_x = b, q_x = a, \mu_x = d, \zeta_x = c$$
$$\mu = e, \xi = f, g = \left(p^2\right), h = \left(q^2\right)$$

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$$Kz_t + Lz_x = \nabla_z S(z)$$

is multi-symplectic in nature with the state variables

$$z = \left(p, q, b, a, \mu, \xi, d, c, e, f, g, h, p^2, q^2\right)^T \in R^{14}$$

The system is multi-symplectic in the sense that K is a skew-symmetric matrix representative of the t direction and L is a skew-symmetric matrix representative of the x direction. S represents a Hamiltonian function

$$q_t - p_{xx} = (\xi q - p\mu)\ ;\ p_t + q_{xx} = (p\xi + q\mu)\ ;\ \ p_x = b\ ;\ q_x = a$$

$$\mu_{tt} - \mu_{xx} - \left((p)^2_{xx} + (q)^2_{xx}\right) = 0$$

$$\xi_{tt} - \xi_{xx} = 0\ ;\ \mu_x = d\ ;\ \xi_x = c;\ \mu_t = e\ ;\ \xi_t = f$$

$$\left(p^2\right)_x = g;\ \left(q^2\right)_x = h$$

and we get

$$q_t - b_x = (\xi q - p\mu)\ ;\ p_t + a_x = (p\xi + q\mu)$$

$$p_x = b;\ q_x = a\ ;\ e_t - d_x - (g_x + h_x) = 0\ ;\ f_t - c_x = 0$$

$$\mu_x = d\ ;\ \xi_x = c\ ;\ \mu_t = e\ ;\ \xi_t = f\ ;\ \left(p^2\right)_x = g;\ \left(q^2\right)_x = h$$

So that

$$\nabla_z S(z) = \left(kp, kq, b, a, sp, sq, d, c, e, f, g, h, p^2, q^2\right)^T$$

$$kp = (\xi q - p\mu)\ ;\ \ kq = (p\xi + q\mu);\ p_x = b\ ;\ q_x = a\ ;$$

$$sp = 0\ ;\ \ sq = 0;\ \mu_x = d\ ;\ \xi_x = c;\ \mu_t = e;$$

$$\xi_t = f\ ;\ \left(p^2\right)_x = g\ ;\ \left(q^2\right)_x = h$$

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$$=\begin{bmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
\quad L=\begin{bmatrix}
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&-1&-1&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

Using midpoint difference scheme to discretize multi-symplectic CNLS system, we can get

$$\frac{q_{l+1/2}^{n+1}-q_{l+1/2}^{n}}{\Delta t}-\frac{b_{l+1}^{n+1/2}-b_{l}^{n+1/2}}{\Delta x}=\left(\hat{\xi}\hat{q}-\hat{p}\hat{\mu}\right) \tag{1}$$

$$\frac{p_{l+1/2}^{n+1}-p_{l+1/2}^{n}}{\Delta t}+\frac{a_{l+1}^{n+1/2}-a_{l}^{n+1/2}}{\Delta x}=\left(\hat{p}\hat{\xi}+\hat{q}\hat{\mu}\right) \tag{2}$$

$$\frac{p_{l+1}^{n+1/2}-p_{l}^{n+1/2}}{\Delta x}=b_{l+1/2}^{n+1/2} \tag{3}$$

$$\frac{q_{l+1}^{n+1/2}-q_{l}^{n+1/2}}{\Delta x}=a_{l+1/2}^{n+1/2} \tag{4}$$

$$\frac{e_{l+1/2}^{n+1}-e_{l+1/2}^{n}}{\Delta t}-\frac{d_{l+1}^{n+1/2}-d_{l}^{n+1/2}}{\Delta x}-\left(\frac{g_{l+1}^{n+1/2}-g_{l}^{n+1/2}}{\Delta x}+\frac{h_{l+1}^{n+1/2}-h_{l}^{n+1/2}}{\Delta x}\right)=0 \tag{5}$$

$$\frac{f_{l+1/2}^{n+1}-f_{l+1/2}^{n}}{\Delta t}=\frac{c_{l+1}^{n+1/2}-c_{l}^{n+1/2}}{\Delta x} \tag{6}$$

$$\frac{\mu_{l+1}^{n+1/2}-\mu_{l}^{n+1/2}}{\Delta x}=d_{l+1/2}^{n+1/2} \tag{7}$$

$$\frac{\xi_{l+1}^{n+1/2}-\xi_{l}^{n+1/2}}{\Delta x}=c_{l+1/2}^{n+1/2} \tag{8}$$

$$\frac{\mu_{l+1/2}^{n+1}-\mu_{l+1/2}^{n}}{\Delta t}=e_{l+1/2}^{n+1/2} \tag{9}$$

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$$\frac{\xi_{l+1/2}^{n+1}-\xi_{l+1/2}^{n}}{\Delta t}=f_{l+1/2}^{n+1/2} \tag{10}$$

$$\frac{(p^2)_{+1/2}^{n+1}-(p^2)_{l+1/2}^{n}}{\Delta x}=g_{l+1/2}^{n+1/2}$$

$$\frac{(q^2)_{+1/2}^{n+1}-(q^2)_{l+1/2}^{n}}{\Delta x}=h_{l+1/2}^{n+1/2}$$

$$\hat{\mu}=\mu_{l+1/2}^{n+1/2},\ \hat{q}=q_{l+1/2}^{n+1/2},\ \hat{p}=p_{l+1/2}^{n+1/2},\ \hat{\xi}=\xi_{l+1/2}^{n+1/2}$$

From (1), (3), we eliminate *b*. So we can get

$$\frac{q_{l+1/2}^{n+1}-q_{l+1/2}^{n}+q_{l+3/2}^{n+}-q_{l+3/2}^{n}}{\Delta t}-\frac{2\left(p_{l+2}^{n+1/2}-2p_{l+1}^{n+1/2}+p_{l}^{n+1/2}\right)}{(\Delta x)^2} \tag{11}$$
$$=\left(\xi_{l+1/2}^{n+1/2}q_{l+1/2}^{n+1/2}-\mu_{l+1/2}^{n+1/2}p_{l+1/2}^{n+1/2}\right)+\left(\xi_{l+3/2}^{n+1/2}q_{l+3/2}^{n+1/2}-\eta_{l+3/2}^{n+1/2}p_{l+3/2}^{n+1/2}\right)$$

From (2), (4), we eliminate *a*. So we can get

$$\frac{p_{l+1/2}^{n+1}-p_{l+1/2}^{n}+p_{l+3/2}^{n+}-p_{l+3/2}^{n}}{\Delta t}+\frac{2\left(q_{l+2}^{n+1/2}-2q_{l+1}^{n+1/2}+q_{l}^{n+1/2}\right)}{(\Delta x)^2}$$

$$=\left(\xi_{l+1/2}^{n+1/2}p_{l+1/2}^{n+1/2}+\mu_{l+1/2}^{n+1/2}q_{l+1/2}^{n+1/2}\right)+\left(\xi_{l+3/2}^{n+1/2}p_{l+3/2}^{n+1/2}+\mu_{l+3/2}^{n+1/2}q_{l+3/2}^{n+1/2}\right) \tag{12}$$

we eliminate e and d. Also g&h. So we can get

$$\left(\frac{\mu_{l+1}^{n+2}-2\mu_{l+1}^{n+1}+\mu_{l+1}^{n}}{(\Delta t)^2}\right)-\left(\frac{\mu_{l+2}^{n+1}-2\mu_{l+1}^{n+1}+\mu_{l}^{n+1}}{(\Delta x)^2}\right)-\left(\frac{(p^2)_{l+2}^{n+1}-2(p^2)_{l+1}^{n+1}+(p^2)_{l}^{n+1}}{(\Delta x)^2}+\frac{(q^2)_{l+2}^{n+1}-(q^2)_{l+1}^{n+1}+(q^2)_{l}^{n+1}}{(\Delta x)^2}\right)=0 \tag{13}$$

we eliminate c and f so we can get

$$\left(\frac{\xi_{l+1}^{n+2}-2\xi_{l+1}^{n+1}+\xi_{l+1}^{n}}{(\Delta t)^2}\right)-\left(\frac{\xi_{l+2}^{n+1}-2\xi_{l+1}^{n+1}+\xi_{l}^{n+1}}{(\Delta x)^2}\right)=0 \tag{14}$$

Multiply (12) with *i.* And adding Eq. (11) then we can get

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$$i\frac{\left(E^{*\,n+1}_{l+2}+2E^{*\,n+1}_{l+1}+E^{*\,n+1}_{l}\right)+\left(E^{*\,n}_{l+2}+2E^{*\,n}_{l+1}+E^{*\,n}_{l}\right)}{2\Delta t}-\frac{\left(E^{*\,n}_{l+2}+E^{*\,n+1}_{l+2}\right)-2\left(E^{*\,n}_{l+1}+E^{*\,n+1}_{l+1}\right)+\left(E^{*\,n}_{l}+E^{*\,n+1}_{l}\right)}{(\Delta x)^2}$$
$$=-\frac{1}{4}\left(\left(E^{*}_{l+1/2}{}^{n}+E^{*}_{l+1/2}{}^{n+1}\right)\left(\eta^{*}_{l+1/2}{}^{n}+\eta^{*}_{l+1/2}{}^{n+1}\right)+\left(E^{*}_{l+3/2}{}^{n}+E^{*}_{l+3/2}{}^{n+1}\right)\left(\eta^{*}_{l+3/2}{}^{n}+\eta^{*}_{l+3/2}{}^{n+1}\right)\right) \quad (15)$$

Conjugating Eq. (15)

$$i\frac{\left(E^{n+1}_{l+2}+2E^{n+1}_{l+1}+E^{n+1}_{l}\right)-\left(E^{n}_{l+2}+2E^{n}_{l+1}+E^{n}_{l}\right)}{2\Delta t}+\frac{\left(E^{n}_{l+2}+E^{n+1}_{l+2}\right)-2\left(E^{n}_{l+1}+E^{n+1}_{l+1}\right)+\left(E^{n}_{l}+E^{n+1}_{l}\right)}{(\Delta x)^2}$$
$$=\frac{1}{4}\left(\left(E^{n}_{l+1/2}+E^{n+1}_{l+1/2}\right)\left(\eta^{n}_{l+1/2}+\eta^{n+1}_{l+1/2}\right)+\left(E^{n}_{l+3/2}+E^{n}_{l+1+3/2}\right)\left(\eta^{n}_{l+3/2}+\eta^{n+1}_{l+31/2}\right)\right)$$

(16)

Multiply (14) with i. And adding Eq. (13) then we can get

$$\frac{\left(\eta^{n+2}_{l+1}-2\eta^{n+1}_{l+1}+\eta^{n}_{l+1}\right)}{(\Delta t)^2}-\frac{\left(\eta^{n+1}_{l+2}-2\eta^{n+1}_{l+1}+\eta^{n+1}_{l}\right)}{(\Delta x)^2}-\left(\frac{\left|E^{n+1}_{l+2}\right|^2-2\left|E^{n+1}_{l+1}\right|^2+\left|E^{n+1}_{l}\right|^2}{(\Delta x)^2}\right)=0 \quad (17)$$

NUMERICAL SIMULATION

In this section, we present the numerical result of the CNLS system using the multi-symplectic integrator. As for conserving quantities, we focus on monitoring the \energy conserving properties of the multi-symplectic integrator.

Now we consider the CNLS system

$$iu_t+u_{xx}+\left(|u|^2+\beta|v|^2\right)u=0$$
$$iv_t+v_{xx}+\left(|v|^2+\beta|u|^2\right)v=0,$$

with the initial value

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From [11,12], we know, when $\beta = 1$ and $\beta = 0$, the CNLS system is the integrable system. Here we consider the interaction of two waves with the initial condition (25). We take the time step $\Delta t = 0.02$ and a space step $\Delta x = 0.2$, $-30 \le x \le 30$, $D_0 = 25$, $r_1 = r_2 = 1$ and $V_0=1$. In Fig. 1, the computation is done for $0 \le t \le 48$. We can see after the colliding of the two soliton waves, they move forward in the same direction and with the same velocity as before

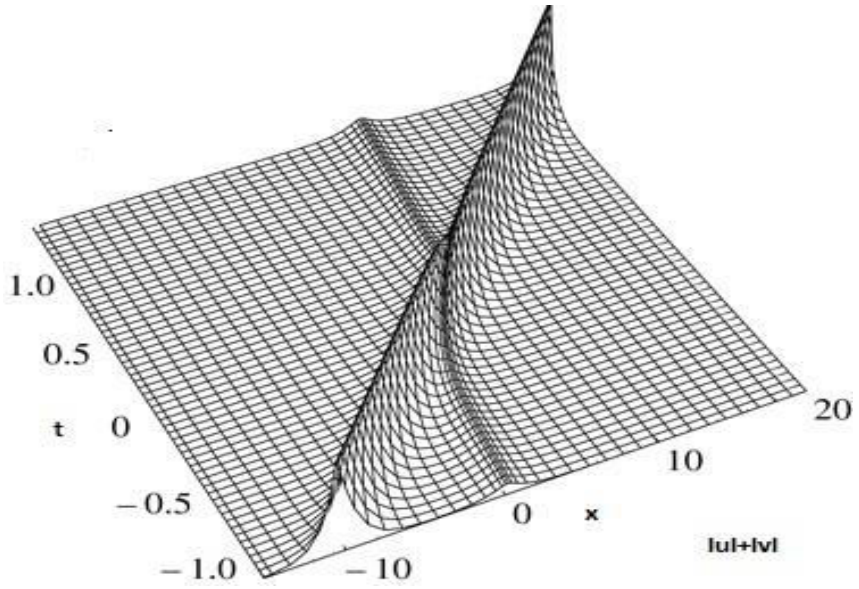


Fig. 2 Simulation results of the interaction of the two waves with β = 1

CONCLUSIONS

The multi-symplectic formulation for the linked 1D nonlinear Schrödinger system is described in this study. There is also discussion of numerical experiments. We find that the multi-symplectic scheme accurately models the development of solitons while preserving energy conservation. It has advantages in terms of long-term computing precision and energy conservation.

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