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A Random Algebraic Polynomial's Asymptotic Estimate of the Number of Level Crossings

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Abstract

The predicted number of level crossings of an algebraic polynomial of degree n with independent distributed random variables as coefficients is investigated. We offer a formula for the anticipated number that has the virtue of being numerically simple. This method demonstrates that the error term in the asymptotic estimate for the anticipated number of real zeros with this type of distribution for the coefficients is equal to zero (1).

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INTRODUCTION

Let us consider a fixed probability space Ω, A, P, and let $a_j(w), j = 0,1.......$ be a sequence of independent random variables defined on Ω. Denote by $N_n(a,b)= N(a,b)$ the number of real zeros of P(x) in the interval $a \le 1 \le b$ where

$$P_n(x,\omega) = P(x) = \sum_{j=0}^{n} a_j(w)x^j \qquad (1.1)$$

Assuming normal distribution with mean zero for the coefficients a_j, J=1…n. **Kac [6]** showed that for all sufficiently large n the mathematical implication of $N(-\infty,\infty)$, denoted by EN, is asymptotically equal to (2/π) of (n+1). There are several authors who have since considered the assumption of normality distributions for the coefficients, work whenever with different conditions for the coefficients or various types of polynomials. For instance, **Sambandhan [8]** showed that the above asymptotic formula remains invariant when the coefficients are dependent with correlation coefficient of any a_j and a_j, I,j=0,1…n as p_{ij}=p $0 < p < \frac{1}{2}, i \neq j$.

However, where p_{ij}=p $0 < p < \frac{1}{2}, i \neq j$. is fixed, the asymptotic formula is reduced by half **FARHAMAND** [4]. This reduction is considered with the work of **IBRAGIMOV AND MASLOVA [5],** when they assume that coefficients a_j have a non-zero mean.**. BHARUCHA AND SAMBANDHAN** [1] improved Kac's estimates for the case of the independent standard normal random coefficients and proved that the error term is 0(1). The lower bound improved by **SAMBANDHAM [7].** Very recently **DAS** [2] published an interesting paper involving several new methods, proved the upper bound further and showed the existence of numerical values of the first six coefficients in, an asymptotic expansion. In particular he obtained

$$EN \sim (2/\pi)\log(n+1) + \sum_{j=0}^{5} D_j(n+1) + 0(n+1)^{-6}.$$

The values of the constants, D_j, j=0,1……5, are numerically evaluate and, unexpectedly, it is observed that D_j=0 for j=1,3,5.

However, the problem is of a different level of difficulty when normality condition is relaxed. In this connection, the case when the belong to the domain of attraction of the normal law with means zero. Prob $(a_j \neq 0)$>0, has been considered by **IBRAGIMOV AND MASLOVA [7]**. They have shown that, for n sufficiently large E $N(-\infty,\infty) \sim (2/\pi)(n+1)$. **DUNNAGE [3]** shown that the no of roots of the same polynomial are approximately E $N(-\infty,\infty) \sim (2/\pi)(n+1)$. Therefore, for this case also the expected number of real zero remains as in the case of Kac referred to above. Hence, it is of special interest to seek a class of distributions, if it exists, such that one can obtain a different number of real zeros for the polynomial. To this **LITTLE WOOD AND OFFORD** [8] have shown that, when the coefficients a_j are independent with a common Cauchy distribution, for all sufficiently large n.

$$EN(-\infty,\infty) \sim C\log(\mathrm{n}+1),$$

where

$$\mathrm{C} = 8\pi^{-2}\int_0^x (xe^{-x})/(x-1+2e^{-x})dx.$$

This shows, since C≈0.741284 and 2/π=0.636620, that for the Cauchy distribution there are asymptotically more zeros. This is probably cause by the fact that in the Cauchy case the variance is infinite and therefore coefficients tend to be more spread out, and hence cancellation is easier.

In this paper we re-examine case and prove the following theorem.

THEOREM

Let

$$\alpha_p = \alpha(p,x) = \sum_{j=0}^{p} x^j, \beta_p = \beta(p,x) = \sum_{j=0}^{p} jx^j \qquad (1.3)$$

$$\gamma_{\mathrm{mn}} = (\alpha_n - 2\alpha_m), \quad \xi_{\mathrm{mn}} = (\beta_n - 2\beta_m), \qquad (1.4)$$

$$I_m = \{(1/\gamma_{\mathrm{mn}})\{[\xi_{\mathrm{mn}} - m\gamma_{mn}]\log[\xi_{\mathrm{mn}} - m\gamma_{mn}] - [\xi_{\mathrm{mn}} - (m+1)\gamma_{mn}]$$
$$\log[\xi_{\mathrm{mn}} - (m+1)\gamma_{mn}] - \gamma_{mn}$$
$$= \sum_{m=0}^{n=1} \mathrm{I}_m$$

and

$$J(x) = (1/\alpha_n)\{\beta n[\log\beta_n - 1] + (\alpha_n n - \beta_n)[\log(\alpha_n n - \beta_n) - 1]\}. \qquad (1.6)$$

Then

$$EN(-\infty,\infty) = 4\pi^{-2}\int_0^1 x^{-1}(I(x) - J(x))dx. \qquad (1.7)$$

As x=0 the integrand in (1.7) has a singularity whose nature must be understood when integrating it numerically. When numerical integration a performed to calculate $EN(0,1) = EN(-\infty,\infty)/4$, it is found that

$EN(0,1) \sim (C/4)\log(n+1) + A_0 + A_2(n+1)^{-2}$ (1.8)

where C is given in (1.2), A_0=0.139783 and A_2=-0.057649.

This result suggests that, for this wider distribution of the coefficients, the error term is also 0(1) and as in **SAMBANDHAN [8]** result, the term of $O(n+1)^{-1}$ vanishes. As the case of Cauchy distributed coefficients studied here has infinite variance and is not well behaved, one can perhaps conjecture that this property of the error term is common to all classes of distribution of coefficients.

A FORMULA FOR THE NUMBER OF REAL ZEROS

The transformations $P(x) \rightarrow P(-x)$ and $P(x) \rightarrow x^n P(1/x)$

Leave the coefficients distribution invariant. Therefore $EN(-\infty,\infty) = 4\,EN(0,1)$, and we confine ourselves to the interval (0,1). By using **Rice [10]** and IBRAGIMOV **and MASLOVA [5]** obtained a formula for a mathematical expectation of the number of real zeros of P(x) as

$$EN(0,1) = \int_0^1 (1/\pi^2 x)dx \left[\int_0^n \log A(u)du + n - n\log A(n) \right.$$

$$\left. + (\beta_n/\alpha_n)\log(n\alpha_n - \beta_n)/\beta_n \right]$$

$$= \pi^{-2}\int_0^1 x^{-1}dx\int_0^n \log g_n(u,x)du,$$

where

$$A(u) = An(u,x) = \sum_{j=0}^{n} |u - j| x^j$$

and

$$g_n(u,x) = \frac{\sum_{j=0}^{n} |(u-j)| x^j}{\sum_{j=0}^{n} (u-j)x^j}$$

Let

$$I(x) = \int_0^n \log\left[\sum_{j=0}^{n} |u-j| x^j\right]dx$$

$$= \int_0^n \log\left[\sum_{j=0}^{n} |u-j| x^j\right]dx$$

$\sin ce\ x \geq 0.$

Now, let m be the greatest contained in u; then, with α_p and β as in (1.3) and γ_{mn} and ζ_{mn} as in (1.4).

$$\sum_{j=0}^{n}|u-j|x^{j} = \sum_{j=0}^{n}|(u-j)|x^{j} + \sum_{j=0}^{n}|(u-j)|x^{j}$$
$$= u(2\alpha_m - \alpha_n) - (2\beta_m - \beta_n),$$
$$= \xi_{mn} - u\gamma_{mn}.$$
$$Thus$$
$$I(x) = \int_0^n \log\left(\xi_{\text{mn}} - u\gamma_{mn}\right)du.$$

Hence the integrand has discontinuous at integral values of u, it is conversant to write

$$I(x) = \sum_{m=0}^{n-1} \mathrm{I}_m$$
$$where$$
$$\mathrm{I}_m = \int_m^{m+1} \log\left(\xi_{mn} - u\gamma_{mn}\right)du,$$

such evaluates to the expression given for I_m in (1.5)

Let

$$J(x) = \int_0^n \log\left|\sum_{j=0}^{n}(u-j)x^{j}\right|du$$
$$= \int_0^n \log|\alpha_n u - \beta_n|du$$

The integrand of J(x) has a singularity at β_n/α_n, which lies between 0 and n after integration, the expression for J(x) in (1.6) is obtained. This result, along with (2.1) and (2.2) establishes (1.7).

NUMERICAL RESULTS

The integration in (1.7) is carried out numerically. As will be seen, a problem arises due to a singularity of the integrand at x=0. The nature of the singularity can be found by considering the behaviour of I(x) and as $x \to 0$. For small values of x, $\alpha_m = 1$, and

$$\gamma_{mn} \cong -1, \beta_0 = 0 \text{ and i} = \text{x, while, if m} > 0, \beta_{\text{m}} \cong x \; and \; \xi_{\text{mn}} = -x,$$

Then, substituting these values into (1.5),

$$I_0 = -x\log x - 1,$$
$$I_m = (m+1)\log(m+1) - m\log m - 1 \qquad (m > 0),$$
$$\sum_{m=1}^{n-1} I_m = n\log n - (n-1)$$
$$J(x) = x\log x + n\log n - n,$$

where $x^{-1}(I(x)-J(x))=-2\log x$. Since the integrand in (1.7) behaves as log x as x→0, he was used to integrate from 0 to 0.98. This routine requires that log x be used a weight function, so it is not appropriate for use up to x=1; the subroutine was used for the remainder of the integration where the integration shows considerable variation on the interval (0.98, 1.0) for large values of n, the degree of the polynomial.

Results have been obtained for a large number of values of n, and a useful selection is presented in Table-1. The final column of the table suggests that Er(n)= EN (0,1)-(C/4)log (n+1) approaches a limit n→∞. To investigate an appropriate behaviour of Er(n) more fully. The values obtained for the coefficients were A_0=0.139782, A_1=0.000065 and A_2=-0.058279. When considering the value of A1, it must be noted that the fact it is non-zero may be due to round off effects. The value found for A1 suggested the use of the lines regression subroutine to fit a line with equation of the form that in (1.8) to the same data, giving

$$Er(n) = 0.139783 - 0.057649(n+1)^{-2}$$

The correlation coefficient was found to be -0.999991, which again must be seen in the context of the rounding off the data. The result obtained in this wider case is consistent with that of KAC [6], when the coefficients are distributed normally, in that he finds the coefficients of (n+1) to be zero.

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