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The Minimum Maximal Domination Energy of a Line Graph

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Abstract

In this paper, the minimum maximal domination energy of a line graph E_DL (G) is introduced. We compute minimum maximal domination energies of some standard graphs and a number of well-known families of graphs. Upper and lower bounds for E_D (G) are established.

MSC: *05C50, 05C69.*

Keywords: *Line graph, Minimum maximal dominating set, minimum maximal domination matrix, minimum maximal eigen values, minimum maximal energy of a line graph.*

INTRODUCTION

In this paper, we are consider a simple graph G = (V,E), that nonempty, finite, have no loops no multiple and directed edges. Let G be such a graph and let n and m be the number of its vertices and edges, respectively. The degree of a vertex v in a graph G, denoted by d(v), is the number of vertices adjacent to v. For any vertex v of a graph G, the open neighborhood of v is the set N(v)={u ϵV: uv ϵ E(G) }. Let G = (V,E) be a graph with E ≠ ϕ. Then E can be thought of as a family of 2 elements subsets of V. the intersection graph Ω (E) is called the line graph of G and is denoted by L(G). Thus the point of L(G) are the lines of G and two points in L(G) are adjacent iff the corresponding lines are adjacent in G. We refer the reader to [8] for more graph theoretical analogist. A subset D of vertices set V of G is called a dominating set of G if every vertex of vϵ(V-D) is adjacent to some vertex in D. The concept of maximal domination

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number was introduced by V.R. Kulli et al.[10]. A dominating set D of a graph G is maximal dominating set if V-D is not a dominating set of G. The maximal domination number γ_m(G) of G is the minimum cardinality of a maximal dominating set. Any maximal dominating set with minimum cardinality is called minimum maximal dominating (denoted MMD) set. For more details in domination theory of graphs we refer to [9].

The concept of energy of a graph was introduced by I. Gutman [6] in the year 1978. Let G be a graph with n vertices and m edges and let A = (a_{ij}) be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, assumed in non increasing order, are the eigenvalues of the graph G. As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy E(G) of G is defined to be the sum of the absolute values of the eigenvalues of G,

$$\text{i.e. } E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

For more details on the mathematical aspects of the theory of graph energy see [2,7,13]. The basic properties including various upper and lower bounds for energy of a graph have been established in [12,14], and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [4,5]. In this paper, we are MMD enery of a graph E_D(G). We arre computed MMD energies of some standards graphs and a number of well-known families of graphs. Upper and lower bounds for E_D(G) are established. It is possible that the MMD energy that we are considering in this paper may be have some applications in chemistry as well as in other areas.

Let G be a graph of order n with vertex set V(G)={$v_1, v_2, \ldots, v_n$} and edge set E(G). A dominating set D of a graph G is maximal dominating set if V-D is not a dominating set of G.The maximal domination number γ_m(G) of G is the minimum cardinality of a maximal dominating set in G. Any maximal dominating set with minimum cardinality is called a MMD set . Let D be a MMD set of a graph G. The MMD matrix of G is the $n \times n$ matrix $A_D(G)=(a_{ij})$,where

$$a_{ij} = \begin{cases} 1, if\ v_i v_j \in E; \\ 1, if\ i = j\ and\ v_i \in M; \\ 0, otherwise \end{cases}$$

The characteristics polynomial of A_D(G) is denoted by

$f_n(G, \lambda)$ = det (λI-A_D(G))

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The MMD eigenvalues of the graph G are the eigenvalues of A_D(G). Since A_D(G) is real and symmetric, its eigenvalues are real numbers and wee label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

The MMD energy of G is defined as:

$$E_D(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The minimum maximal domination energy in line graph

Let L(G) be a line graph of order n with vertex set V(L(G))={v1,v2,…,vn} and edge set E(G). A dominating set D of a line graph L(G) is maximal dominating set if V-D is not a dominating set of L(G).The maximal domination number γ_m(L(G)) of L(G) is the minimum cardinality of a maximal dominating set in L(G). Any maximal dominating set with minimum cardinality is called a MMD set . Let D be a MMD set of a line graph L(G). The MMD matrix of L(G) is the $n \times n$ matrix A_D(L(G))=(a_{ij}), where

$$a_{ij} = \begin{cases} 1, if\ v_i v_j \in E; \\ 1, if\ i = j\ and\ v_i \in M; \\ 0, otherwise \end{cases}$$

The characteristics polynomial of A_D(L(G)) is denoted by

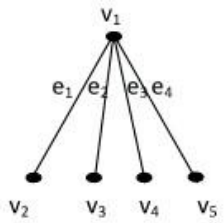
$$f_n(L(G), \lambda) = \det(\lambda I - A_D(L(G)))$$

The MMD eigenvalues of the graph L(G) are the eigenvalues of A_D(L(G)). Since A_D(L(G)) is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The MMD energy of L(G) is defined as:

$$E_D(L(G)) = \sum_{i=1}^{n} |\lambda_i|.$$

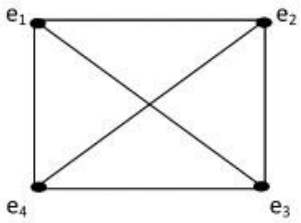
We first compute the minimum maximal dominating energy of L (G)

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G

Figure 1.



L(G)

Figure 2.

Example 3.2.1

Let G be a graph (Fig.1) and L(G) be its line graph (Fig.2) with vertices $V(L(G))=\{e_1, e_2, e_3, e_4\}$ and let its minimum maximal dominating set be $D_1=\{e_1\}$. Then

$$A_{D_1}(L(G)) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

The characteristics polynomial of $A_{D_1}(L(G))$ is

$f_n(L(G),\lambda) = \lambda^4 - \lambda^3 - 6\lambda^2 - 5\lambda - 1$

Hence, the minimum maximal dominating eigenvalues are

$\lambda_1 \approx 1$, $\lambda_2 \approx 1$, $\lambda_3 \approx 3.3028$, $\lambda_4 \approx -2.6056$

Therefore the minimum maximal dominating energy of L(G) is

$E_{D_1}(L(G)) \approx 2.6972$

If we take another minimum maximal dominating set of G, namely $D_2=\{e_2\}$, then

$$A_{D_2}(L(G)) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

The characteristics polynomial of $A_{D_2}(L(G))$ is

$f_n(L(G),\lambda) = \lambda^4 - \lambda^3 - 6\lambda^2 - 5\lambda - 1$

Hence, the minimum maximal dominating eigenvalues are $\lambda_1 \approx 1$, $\lambda_2 \approx 1$, $\lambda_3 \approx 3.3028$, $\lambda_4 \approx -2.6056$.

Therefore the minimum maximal dominating energy of L(G) is

$E_{D_2}(L(G)) \approx 2.6972$

This example illustrate the fact that the minimum maximal dominating energy of a line graph L(G) depends on the choice of the minimum maximal dominating set. i.e. the minimum maximal dominating energy

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is not a graph invariant. In the following section, we introduce some properties of characteristic polynomials of minimum maximal dominating matrix of a line graph L(G).

SOME PROPERTIES OF MMD ENERGY OF LINE GRAPHS

Theorem:

Let L(G) be a graph of order n. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A_D(L(G)). Then

i) $\sum_{i}^{n} \lambda_i = |D|$

ii) $\sum_{i}^{n} \lambda_i^2 = |D| + 2m$

Proof:

i) Since the sum of the eigenvalues of A_D(L(G)) is the trace of A_D(L(G)).

Then

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |D|$$

Similarly the sum of squares eigenvalues of A_D(L(G)) is the trace of $A_D(\mathrm{L(G)})^2$.

Then

$$\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} a_{ji}$$

$$= \sum_{i=1}^{n} a_{ii}^2 + \sum_{i \neq j}^{n} a_{ij} a_{ji}$$

$$= \sum_{i=1}^{n} a_{ii}^2 + 2 \sum_{i<j}^{n} a_{ij}^2$$

$$= |D| + 2m$$

Bapat and S.Pati [3] proved that if the graph energy is a rational number then it is an even integer

Theorem:

Let L(G) be a line graph with a MMD set. If the MMD energy E_D(L(G)) of L(G) is a rational number, then E_D(L(G)) $\equiv |D|$ (mod 2)

Proof:

Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be MMD eigenvalues of a line graph L(G) of which $\lambda_1, \lambda_2, \ldots, \lambda_r$ are positive and the rest are non-positive , then

$\sum_{i=1}^{n} |\lambda_i| = (\lambda_1 + \lambda_2 + \cdots + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} + \cdots + \lambda_n)$

$= 2(\lambda_1 + \lambda_2 + \cdots + \lambda_r) - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)$

$= 2q - |D|$ where $q = \lambda_1 + \lambda_2 + \cdots + \lambda_r$

Therefore, E_D(L(G)) = 2q - $|D|$, and the proof is completed.

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MINIMUM MAXIMAL DOMINATION ENERGY OF SOME STANDARD LINE GRAPHS

Theorem:

For n≥2, the minimum maximal domination of a star graph $K_{1,n-1}$ is (n-2)+$\sqrt{n^2-2n+5}$

Proof:

Let $K_{1,n-1}$ be a star graph with vertex set V = $\{e_1, e_2, e_3, \ldots e_{n-1}\}$, e_1 is the center, and the MMD set is D = $\{e_1\}$

$$A_D(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 \cdots 1 \\ 1 & 0 & 1 \cdots 1 \\ 1 & 1 & 0 \cdots 1 \\ \vdots & \vdots & \vdots \ddots \vdots \\ 1 & 1 & 1 \ldots 0 \end{pmatrix}$$

The characteristic polynomial of $A_D(K_{1,n-1})$ is

$f_n(K_{1,n-1}, \lambda)$ = det (λI-$A_D(K_{1,n-1})$)

$$f_n(K_{1,n-1}, \lambda) = \begin{bmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda & -1 & \cdots & -1 & -1 \\ -1 & -1 & \lambda & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & \lambda \end{bmatrix}$$

$=(\lambda+1)^{n-2}$ (λ^2-(n-1) λ – 1)

By analysis the last factor of $f_n(K_{1,n-1}, \lambda)$ we get

$$f_n(K_{1,n-1}, \lambda) = (\lambda+1)^{n-2} (\lambda^2\text{- (n-1) } \lambda - 1)$$

$$= (\lambda+1)^{n-2} \left(\frac{n-1 \pm \sqrt{n^2-2n+5}}{2}\right)$$

$$= (\lambda+1)^{n-2} + \left(\frac{n-1+\sqrt{n^2-2n+5}}{2}\right) + \left(\frac{n-1-\sqrt{n^2-2n+5}}{2}\right)$$

It follows that the MMD spectrum is

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MMD spec $(K_{1,n-1}) \approx \begin{pmatrix} -1 & \left(\frac{n-1+\sqrt{n^2-2n+5}}{2}\right) & \left(\frac{n-1-\sqrt{n^2-2n+5}}{2}\right) \\ n-2 & 1 & 1 \end{pmatrix}$

$= (n-2) + \sqrt{n^2-2n+5}$

Therefore, the MMD energy of a star graph

$E_D(K_{1,n-1}) \leq (n-2) + \sqrt{n^2-2n+5}$

Theorem:

For path P_n, $n \geq 3$, then

i) n=3, $E_D(P_3) = \sqrt{5}$

ii) n=4,

iii) n=5,

Proof:

i) Let P_n be a path with vertex set V(L(G)) = $\{e_1, e_2, \ldots e_n\}$. The minimum maximal dominating set is D = $\{e_1\}$. Then

$A_D(P_3) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

The respective characteristic polynomial is

$f_n(P_3, \lambda) = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{vmatrix}$

$= \lambda^2 - \lambda - 1$

The minimum maximal dominating spectrum of P_3 will be written as

MMD spec$(P_3) = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$

$= \sqrt{5}$

Therefore, the minimum maximal dominating energy of a path

$E_D(P_3) = \sqrt{5}$

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i) Let P_n be a path with vertex set V(L(G)) = $\{e_1, e_2, \ldots e_n\}$. The minimum maximal dominating set is D = $\{e_2\}$. Then $A_D(P_4)$ = $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

The respective characteristic polynomial is

$$f_n(P_4, \lambda) = \begin{vmatrix} \lambda & -1 & 0 \\ -1 & \lambda - 1 & -1 \\ 0 & -1 & \lambda \end{vmatrix}$$

$= \lambda^3 - \lambda^2 - 2\lambda$

The minimum maximal dominating spectrum of P_4 will be written as

$$\text{MMD spec}(P_4) = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = 1$$

Therefore, the minimum maximal dominating energy of a path

$E_D(P_4) = 1$

ii) Let P_n be a path with vertex set V (L(G)) = $\{e_1, e_2, \ldots e_n\}$. The minimum maximal dominating set is D = $\{e_2, e_3\}$. Then

$$A_D(P_5) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The respective characteristic polynomial is

$$f_n(P_5, \lambda) = \begin{vmatrix} \lambda & -1 & 0 & 0 \\ -1 & \lambda - 1 & -1 & 0 \\ 0 & -1 & \lambda - 1 & -1 \\ 0 & 0 & -1 & \lambda \end{vmatrix}$$

$= \lambda^4 - 2\lambda^3 - 2\lambda^2 - 2\lambda + 1$

The minimum maximal dominating spectrum of P_4 will be written as

$$\text{MMD spec}(P_4) = \begin{pmatrix} 1 & -1 & \frac{2+\sqrt{8}}{2} & \frac{2-\sqrt{8}}{2} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Therefore, the minimum maximal dominating energy of a path

$E_D(P_5) = 2 + \sqrt{8}$

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