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Expected Number of Roots of a Random Trigonometric Polynomial

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Abstract

In this paper we estimate the number of roots of a random trigonometric polynomial of degree n with independent normally distributed random real coefficients asymptotically crosses the line ***mx****, when m is any real value such that* $(m^2/n)\to 0$ *as* $n\to\infty$*. in our work we have proved that for* $\mathbf{m > \exp(nf)}$ *, where f is any function of n such that* $f(n)\to\infty$*, the expected number of level crossings of the above trigonometric polynomial reduces to only one.*

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1. INTRODUCTION

Let $T_n(\theta) = \sum_{k=1}^{n} k^p y_k(w)\cos k\theta \qquad (1)$

where $(y_1(w),\ldots y_n(w))$, $0<w<1$, is a sequence of mutually independent identically distributed random variables lying in the domain of attraction of normal law. Let $N_n(\beta,\gamma)$ be the number of real zeros of $T_n(\theta)=0$ in the interval $\beta \le \theta \le \gamma$ with multiple zeros counted only once. Previously **Dunnage [3]and Farhamand [7]** found that in the case of normally distributed random variables with mean zero and variance one, the above polynomial (1) has $\frac{2n}{\sqrt{3}} + 0\left(n^{11/13}(\log n)^{3/13}\right)$ zeros, on the average, in the interval (0, 2Π), except for a certain exceptional set whose measure

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does not exceed $(\log n)^{-1}$. **Ibragimov and Maslova [3]** considered a corresponding algebraic equation whose coefficients $y_j(w)$'s (j=1,2,….n) were identically distributed random variables belonging to the domain of attraction of normal law with p=0, mean zero and $P(y_j \neq 0) > 0$; and found that the average number of real zeros of the curve is asymptotic to (2/Π) log n. **Little wood and Offord [7]** also consider the same polynomial with mean zero and variance one and found as similar to **Dunnage [3] .** The objective of this paper is to find an estimate for upper bound of the average number of real zeros of the polynomial (1) in the interval (0, 2/Π) when $y_j(w)$'s belong to domain of attraction of normal law.

For this we divide $N_n(0, 2/\Pi)$ into two groups, taking $\in = \frac{1}{\sqrt{n}}$;

(i). those lying in the intervals $(0, \in), (\pi - \in, \pi + \in)$ and $(2\pi - \in, 2\pi)$,

(ii) those lying in the intervals $(\in, \pi - \in)$ and $(\pi + \in, 2\pi - \in)$.

The zeros in group (i) intervals are less than the zeros in group (ii) intervals. This we prove in lemma-2. Zeros which make significant contribution to the final result are those lying in group (ii) intervals.

Theorem :- Let $T_n(\theta) = \sum_{k=1}^{n} k^p y_k(w) \cos k\theta$ be a random trigonometric polynomial, where the coefficients $y_k(w)$'s (k=1,2,……n) are identically distributed independent random variables lying in the domain of attraction of normal law with characteristic function $\exp\left\{-\frac{t^2}{2}h(t)\right\}$ and $\exp\left\{-\frac{t^2}{2}H(t)\right\}$ in the neighborhood of zero and infinity respectively, where h(t) and H(t) are positive slowly varying functions as $t \to 0$ *and* $t \to \infty$. Then, we obtain that an upper estimate for the average number of real zeros of $T_n(\theta)$ in the interval (0,2Π) is equal to

$$2n\left(\frac{2p+1}{2p+3}\right)^{1/2}\left(1+\sqrt{\log n}\right)^n + 0\left(\sqrt{n}\right),$$

when n is sufficiently large.

1.2. Some Lemmas

In this section we establish two lemmas which will be used in the proof of the theorem.

Lemma 1: If h(t) be a slowly varying function as $t \to 0$ and H(t) be a slowly varying function as $t \to \infty$, then for $0 < p < 1$, we have

(i) $\lim_{t \to 0} t^p h(t) = 0$ and $\lim_{t \to \infty} t^{-p} h(t) = \infty$.

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(ii) $\lim_{t\to\infty} t^{-p}H(t) = 0$ and $\lim_{t\to\infty} t^{-p}H(t) = \infty.$ (2)

Lemma 2:

If n (e) denotes the number of real zeros of $T_n(z)$ in $|z| \le \in$, then

$$P\left[n(\in) > (\log 2)^{-1}(\log n + 4n \in + \log D_n)\right] \le \frac{3}{n^{1-\rho}},$$

for $0 < \rho < 1$, where $D_n = \sum_{k=1}^{n} k^p$

1.3 FORMULA FOR $EN_n(\beta,\gamma)$

The results of this section are based on a discussion of **Kac [5] and Rice [8].** Let $F(y_k)$ be the distribution function and $f(t_k)$ be the characteristic function of the random variable $y_k(w)=y_k$. Then by the Levy's inversion formula for the continuity points y_k and y_{k+h} $(h>0)$ of $F(y_k)$, we have

Let the n-dimensional set of points be denoted by R_n. The probability mass attached to the infinitesimal rectangle $\prod(\bar{y})$, containing the point $\bar{y} = (y_1, y_2, \ldots\ldots y_n)$ is.

$$dp(\bar{y}) = \prod_{k=1}^{n}\left\{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it_k y_k} it_k f(t_k)dt_k\right)dy_k\right\}$$

$$= (2\pi)^{-n}\left\{\left(\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} e^{-i\sum_{k=1}^{n} t_k y_k} f(t_1)f(t_2)\ldots\ldots f(t_n)dt_1..dt_n\right)dy_1 dy_2..dy_n\right\}$$

……(3)

Where $dy_1, dy_2, \ldots\ldots dy_n$ are the lengths of the sides of $\prod(\bar{y})$. Now following the procedure of **Kac [6],** we obtain

$$EN_n(\beta, y) = \frac{1}{2\pi}\int_{\beta}^{\gamma} d\theta \int_{-\infty}^{\infty} d\xi \; Rn\,(\xi,\theta) \qquad (4)$$

Where

$$R_n(\xi,\theta) = (2\pi)^{-n}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\ldots - i\sum_{k=1}^{n} t_k y_k f(t_1)\ldots\ldots f(t_n)dt_1\ldots\ldots dt_n\right).$$

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$$x\cos\left(\xi T_n(\theta)\right)\left|Tn'(\theta)\right|dy_1.....dy_n \qquad (5)$$

(Since ρ>0 is arbitrary, choosing ρ=1/n, we have $n^{1/n}\sim 1$)

Hence

$$\int_{-\infty}^{\infty}\log\left(\frac{\sum_{k=1}^{n}\left|\nu k^{p}\cos k\theta - k^{p+1}\sin k\theta\right|^{2}}{\sum_{k=1}^{n}\left|\nu k^{p}\cos k\theta\right|^{\alpha}}\right)d\nu$$

$$=2\int_{-\infty}^{\infty}\frac{d\eta}{\eta}\int_{0}^{\infty}\left(e^{-\sum_{k=1}^{n}\left(\nu\beta_k\eta\right)^2} - e^{-\sum_{k=1}^{n}\left(\nu\beta_k\eta-\alpha_k\eta\right)^2}\right)d\nu$$

(by using Frullani's theorem again and putting $\beta_k = k^p\cos k\theta$ and $\alpha_k = k^{p+1}\sin k\theta$)

$$=\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}\left(\frac{e^{-X_n\xi^2} - e^{-(X_n\xi^2 - Yn\,\xi\eta + Zn\,\eta 2)}}{\eta^2}\right)d\eta. \qquad (6)$$

(putting $\nu\eta = \xi$, $\mathrm{Xn} = \sum_{k=1}^{n}\beta k2$, $\mathrm{Yn} = 2\sum_{k=1}^{n}\alpha_k\beta_k$ *and* $\mathrm{Zn} = \sum_{k=1}^{n}\alpha k^2$)

From above equations we have

$$EN_n(\beta,\gamma) < \frac{\left(1+\sqrt{\log n}\right)^n}{\pi^2}\int_{\beta}^{\gamma}d\theta\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}\left(\frac{e^{-X_n\xi^2} - e^{-(X_n\xi^2 - Yn\,\xi\eta + Zn\,\eta 2)}}{\eta^2}\right)d\eta. \qquad (7)$$

In order to evaluate

$$I=\int_{-\infty}^{\infty}\left(\frac{e^{-X_n\xi^2} - e^{-(X_n\xi^2 - Yn\,\xi\eta + Zn\,\eta 2)}}{\eta^2}\right)d\eta$$

We apply the method suggested in **Das [3].**

We put $t = Y_n\xi, Z_n = h/2$ to obtain

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$$I = e^{-X_n\xi^2}\int_{-\infty}^{\infty}\frac{1-e^{(t\eta-\frac{1}{2}h\eta 2)}}{\eta^2}d\eta$$

Now we split the last integral into three integrals p(h)+q(h)+r(h), where

$$p(h)=\int_{-\infty}^{\infty}\frac{1-e^{-\frac{1}{2}h\eta^2}}{X^2}dx \qquad (h > 0)$$

$$q(h)=\int_{-\infty}^{\infty}\frac{tx}{X^2}e^{-\frac{1}{2}h\eta^2}dx \qquad (h > 0)$$

and

$$r(h)=-\int_{-\infty}^{\infty}\left(\sum_{\nu=2}^{\infty}\frac{t^\nu x^\nu}{\nu!}\right)\frac{e^{-\frac{1}{2}hx^2}}{X^2}dx \qquad (h > 0)$$

As in Das, we shall have

$$P(h)=\sqrt{2\pi h} and q(h)=0$$

$$r(h)=\sum_{\nu=1}^{\infty}\frac{\sqrt{\pi h\left(\frac{2t^2}{h}\right)^\nu}}{2\nu\,\nu!(2\nu-1)}=\sum_{\nu=1}^{\infty}\frac{\sqrt{2\pi h}}{4\nu\,\nu!(2\nu-1)}\left(\frac{Y_n{}^2\xi}{Z_n}\right)^\nu$$

and

$$\int_{-\infty}^{\infty}Id\xi=\frac{\pi\sqrt{4X_nZ_n-Y_n{}^2}}{X_n} \qquad (8)$$

From above equations we have

$$EN_n(\beta,\gamma)<\frac{\left(1+\sqrt{\log n}\right)^n}{\pi^2}\int_{\beta}^{\gamma}\frac{\sqrt{4X_nZ_n-Y_n{}^2}}{X_n}d\theta. \qquad (9)$$

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1.4. Proof of the Theorem

The set of values of w for which either $4X_n(w)Z_n(w) = Y_n^2(w)$ or $Z_n(w) = 0$ will be called an w-set. In the present case this set is $\left\{0, \pm\frac{\pi}{2}, \pm\pi, ...\right\}$. We take (β, γ) as $(\varepsilon, \pi - \varepsilon)$ and $(\varepsilon, 2\pi - \varepsilon)$. successfully. For the interval $(\varepsilon, \pi - \varepsilon)$ we use estimations derived in the last chapter, i.e.

$$X_n = \frac{n^{2p+1}}{2(2p+1)}\left[1 + 0\left(\frac{1}{n\varepsilon}\right)\right]$$

$$Y_n = 0\left(\frac{n^{2p+1}}{\varepsilon}\right),$$

$$Z_n = \frac{n^{2p+3}}{2(2p+3)}\left[1 + 0\left(\frac{1}{n\varepsilon}\right)\right]$$

and

$$\frac{\sqrt{X_n Z_n - Y_n^2}}{X_n} = \left(\frac{2p+1}{2p+3}\right)^{1/2} n[1 + 0(\varepsilon)].$$

Hence from above equations, we have

$$EN_n(\varepsilon, \pi - \varepsilon) < \frac{\left(1 + \sqrt{\log n}\right)^n}{\pi} \int_{\varepsilon}^{\pi-\varepsilon} \left(\frac{2p+1}{2p+3}\right)^{1/2} n[1 + 0(\varepsilon)].$$

$$\sim \left(\frac{2p+1}{2p+3}\right)^{1/2} n\left(1 + \sqrt{\log n}\right)^n. \quad (10)$$

Similarly we can prove that

$$EN_n(\pi + \varepsilon, 2\pi - \varepsilon) < \left(\frac{2p+1}{2p+3}\right)^{1/2} n\left(1 + \sqrt{\log n}\right)^n. \text{(for large n)} \quad (11)$$

From lemma-2 it is evident that the number of zeros in the interval $(-\varepsilon, \varepsilon)$ is $0(n^{1/2})$. By periodicity the number of zeros in $(-\varepsilon, \varepsilon)$. Arguments similar to those in Lemma-2 prove that the number of zeros in the interval $(\pi - \varepsilon, \pi + \varepsilon)$ is also $0(n^{1/2})$. These observations together with (11) proves the theorem.

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