

Monte Carlo Estimation: Computing Quantities by Random Sampling and Its Slow but Sure Convergence

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ABSTRACT

Some quantities, such as the value of a complicated integral or the behaviour of a complex system, are hard to compute exactly but can be estimated by a surprising method: generating many random samples and averaging the result, the Monte Carlo method; the estimate it gives is not exact but improves as more samples are used, converging to the true value slowly but surely, and in a way that does not worsen with the complexity of the problem. This study examines Monte Carlo estimation, computing quantities by random sampling, and its slow but sure convergence. The estimate of a known quantity, obtained by random sampling, was examined as the number of samples was increased, so that how the estimate converged to the true value, and how fast, could be seen. The estimate fluctuated around the true value and settled toward it as the number of samples grew, the fluctuations shrinking but slowly: with few samples the estimate was rough and varied considerably from run to run, and as samples were added it homed in on the true value, but the error fell only in proportion to one over the square root of the number of samples, so that reducing the error tenfold required a hundredfold more samples. The convergence was therefore sure but slow, each further digit of accuracy costing a hundredfold more work. The method's strength was that this rate did not worsen as the problem grew more complex or higher-dimensional, unlike methods that divide the problem into a grid, whose cost explodes with dimension. Monte Carlo is therefore valued not for speed on simple problems, where it is slow, but for problems too complex or high-dimensional for other methods, where its dimension-independent convergence makes it the practical choice. The study shows that Monte Carlo estimation converges to the true value as one over the square root of the number of samples, and it discusses the implications for computation in engineering and statistics.

KEYWORDS: *Monte Carlo method, Random sampling, Estimation, Convergence, Square-Root law, Variance reduction, High-dimensional integration, Simulation, Numerical methods*

INTRODUCTION

Some quantities are easy to define but hard to compute. A complicated integral over many variables, the average behaviour of a system with many interacting parts, the probability of a rare combination of events, may have no formula and resist the usual numerical methods. For such problems there is a method that seems almost paradoxical: instead of computing the quantity directly, one generates many random samples and averages a result computed from each, and the average estimates the quantity. This is the Monte Carlo method (1, 2).

The idea works because many quantities can be expressed as an average, an expectation, over some randomness, and an average can be estimated by sampling: just as the average height of a population can be estimated from a random sample of people, so the value of an integral or the behaviour of a system can be estimated from random samples drawn appropriately. The estimate is not exact, being based on a finite random sample, but it is unbiased, centred on the true value, and it improves as more samples are taken (2, 3).

How it improves, and how fast, is the crux. The estimate converges to the true value as the number of samples grows, but slowly, and understanding the rate is essential to knowing when Monte Carlo is worthwhile. This paper examines Monte Carlo estimation, computing quantities by random sampling, and its slow but sure convergence, examining the estimate of a known quantity as the number of samples is increased. The aim is to show how the estimate converges and how fast, and to discuss the implications for computation in engineering and statistics.

LITERATURE REVIEW

The Monte Carlo method, estimating quantities by random sampling, was named and developed in the mid-twentieth century for problems intractable by direct computation, and has grown into a broad family of techniques set out in the standard texts (1, 2, 3). A central result is that the error of a Monte Carlo estimate falls in proportion to one over the square root of the number of samples, a slow but dimension-independent rate (2, 4).

CONVERGENCE AND ITS RATE

The literature describes the method and its convergence:

- Many quantities are averages that can be estimated by random sampling (2, 3).
- The estimate is unbiased and converges to the true value as samples grow (2).
- Its error falls in proportion to one over the square root of the number of samples (2, 4).
- This rate does not worsen with the dimension of the problem (2, 7).

A recurring theme is convergence that is slow but sure and independent of dimension.

The literature establishes the square-Root law: because the estimate is an average of random samples, its error, by the behaviour of such averages, falls in proportion to one over the square root of the number of samples, so that a tenfold reduction in error costs a hundredfold more samples (2, 4). It emphasises that this rate is independent of the dimension of the problem, which is the method's great advantage over grid-based methods whose cost rises explosively with dimension, and it develops variance-reduction techniques that lower the error for a given number of samples, throughout the texts on Monte Carlo and simulation (3, 5, 6, 7, 8, 9, 10).

RESEARCH GAP

Although the convergence of Monte Carlo is well understood, a concise demonstration that shows an estimate converging to the true value as samples grow, bringing out the slow square-Root rate, remains useful as a clear illustration. There is therefore value in a study that displays the estimate against the number of samples and relates its convergence to when Monte Carlo is worthwhile.

OBJECTIVES

The objectives of this study are:

- To examine the estimate of a known quantity by random sampling as the number of samples grows.
- To show how the estimate converges to the true value, with shrinking fluctuations.
- To show how the error falls in proportion to one over the square root of the number of samples.
- To relate the slow but dimension-independent convergence to when Monte Carlo is worthwhile.

METHODOLOGY

1. Estimate Against Number of Samples

The estimate of a known quantity, obtained by random sampling, was examined as the number of samples was increased, the estimate being the average of a result computed from each sample, and converging to the true value as the samples grew (2). The elements are summarised in Table 1.

Table 1: Elements of Monte Carlo estimation

Element	Description
Random samples	Random inputs drawn for the estimate
Estimate	Average of results over the samples
True value	The quantity being estimated
Error	Difference between estimate and true value

2. Observing the Convergence

The estimate was examined as the number of samples grew, its fluctuations about the true value shrinking, and the error tracked, falling in proportion to one over the square root of the number of samples (2, 4). The independence of this rate from the dimension of the problem was noted (7).

RESULTS AND FINDINGS

The estimate fluctuated around the true value and settled toward it as the number of samples grew, the fluctuations shrinking but slowly, as shown in Figure 1, which plots the estimate against the number of samples. With few samples the estimate was rough and varied considerably from run to run, and as samples were added it homed in on the true value, but the error fell only in proportion to one over the square root of the number of samples, so that reducing the error tenfold required a hundredfold more samples (2, 4).

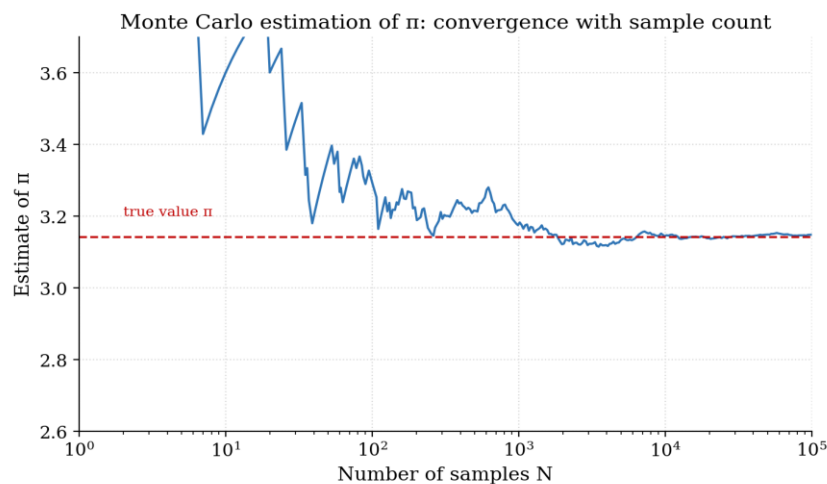


Figure 1: Monte Carlo estimation of π : convergence with sample count

The convergence was therefore sure but slow, each further digit of accuracy costing a hundredfold more work. The method's strength was that this rate did not worsen as the problem grew more complex or higher-dimensional, unlike methods that divide the problem into a grid, whose cost explodes with dimension (7). The findings are summarised in Table 2.

Table 2: Estimate and error with number of samples

Number of samples	Estimate	Error
Few	Rough, variable	Large
Many	Close to true value	Small
Hundredfold more	—	Tenfold smaller
Higher dimension	Same rate	Unchanged by dimension

DISCUSSION

The results show that Monte Carlo estimation converges to the true value as one over the square root of the number of samples, and both the sureness and the slowness follow from its being an average. Because the estimate is the average of many independent random results, it is governed by the laws of averages: as more results are averaged, their fluctuations cancel and the average settles toward the true expected value, so convergence is sure, the estimate approaching the truth as the samples grow without bound. But the spread of an average falls

only as one over the square root of the number averaged, so the error of the estimate falls at that rate too; this is why the convergence, though sure, is slow, and why each tenfold gain in accuracy costs a hundredfold more samples (2, 4).

This slowness means Monte Carlo is a poor choice where other methods work, which must be acknowledged. For a simple one-dimensional integral, a quadrature rule reaches high accuracy with few evaluations, its error falling steeply with the number of intervals, whereas Monte Carlo would need vast numbers of samples for the same accuracy; on such problems Monte Carlo is far slower and not the tool to use. Its square-Root rate is humble compared with the fast convergence of good deterministic methods on well-behaved low-dimensional problems (4). If speed on simple problems were the criterion, Monte Carlo would rarely be chosen.

Its decisive advantage is that the square-Root rate does not worsen with dimension, which is the key practical lesson and the reason for its use. Grid-based methods divide each dimension into intervals, so the number of points, and hence the work, rises as a power of the dimension: a grid fine enough in one dimension becomes astronomically large in ten or a hundred, the so-called curse of dimensionality, making such methods hopeless for high-dimensional problems. Monte Carlo's error, by contrast, depends only on the number of samples, not on the dimension, so its cost to reach a given accuracy does not explode as dimensions are added; in high dimensions it is often the only feasible method, and its slow rate is then beside the point, since the alternatives cannot run at all (2, 7). This is why Monte Carlo is the workhorse for high-dimensional integrals and complex stochastic systems in physics, engineering, and finance (9). Where its slowness bites, it can be mitigated by variance-reduction techniques, which lower the error for a given number of samples by sampling more cleverly, though they do not change the underlying square-Root rate (3, 8). The practical conclusion is that Monte Carlo is chosen not for speed on simple problems, where it is slow, but for complex and high-dimensional ones, where its dimension-independent convergence makes it feasible, the main caveats being its slow rate and reliance on good random sampling, and that the figure illustrates the convergence for one estimate rather than all.

CONCLUSION

Monte Carlo estimation, computing quantities by random sampling, and its convergence were examined by examining the estimate of a known quantity as the number of samples grew. The

estimate fluctuated around the true value and settled toward it as samples were added, but the error fell only in proportion to one over the square root of the number of samples, so a tenfold reduction in error cost a hundredfold more samples, a convergence sure but slow; crucially, this rate did not worsen with the dimension of the problem. The study shows that Monte Carlo estimation converges as one over the square root of the number of samples, so that it is chosen not for speed on simple problems, where it is slow, but for complex and high-dimensional ones, where its dimension-independent convergence makes it the practical choice, with the caveats of its slow rate and reliance on good random sampling and that the figure illustrates the convergence for one estimate rather than all.

LIMITATIONS

The following limitations apply to this study:

- Convergence is slow, so high accuracy requires very many samples.
- The figure illustrates the convergence for one estimate rather than for all problems.
- The method relies on good random sampling; poor randomness biases the estimate.
- Variance-reduction techniques help but do not change the square-Root rate.

FUTURE SCOPE

Future work could extend the present study as follows:

- Variance-reduction techniques and the accuracy they buy for a given number of samples.
- Comparison of Monte Carlo with quadrature across dimensions to locate the crossover.
- Quasi-Monte Carlo methods and their improved convergence.
- Monte Carlo for high-dimensional integrals and complex stochastic systems.
- The effect of the quality of random number generation on the estimate.

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